Abstract—This paper proposes a new approach based on the unknown input method to synthesize the observer for polynomial Takagi-Sugeno (T-S) fuzzy system with uncertainties. In this paper, the upper bounds of uncertainties are not given and the effect of uncertainties is eliminated without designing an extra controller. With the aids of the non-common Lyapunov theory and Matlab’s tools of the Sum-of-Square (SOS), a new observer is synthesized in which the observer form is completely different from the traditional observer forms reported in previous papers. The conditions for the observer synthesis are much relaxed and the complexity of the design process is reduced. Finally, two illustrative examples are presented to demonstrate the effectiveness of the proposed method.

Index Terms—Uncertain polynomial T-S fuzzy systems, observer synthesis, unknown inputs, Sum of Square (SOS).

I. INTRODUCTION

Takagi-Sugeno (T-S) fuzzy model [1]-[4] has received a great deal of attention in control system research area. This model has provided another way to solve the problems of nonlinear control systems. Therefore, many theories and control design methods for linear control systems can be applied in the T-S fuzzy system. In addition, a new approach for designing an optimal coordination controller based on the adaptive fuzzy dynamic programming and game theory for solving the consensus problem of multi-agent differential games was studied in [5]. There was a book [6] to study the Type-2 fuzzy logic in detail and a related study to design an adaptive slide mode controller for the interval Type-2 fuzzy systems was reported in [7]. In 2009, a more general form of the T-S fuzzy model called the polynomial T-S fuzzy model has been introduced in [8] and an interval Type 2 polynomial fuzzy model was investigated in [9]. This model allows the system matrices containing polynomial forms in its entries instead of only constant forms. With the supporting Sum-Of-Square (SOS) Tools in Matlab [10], the polynomial T-S fuzzy system can be considered as an effective tool for modeling nonlinear control systems. Recently, a large number of studies focused on the polynomial T-S fuzzy systems such as controller design, observer design, and stability analysis [8]-[24]. For example, the stability analyses for the polynomial T-S fuzzy systems by employing the multiple Lyapunov function and switching Lyapunov function were investigated in [13] and [14], respectively. Besides, the controller design and observer-based controller design for the polynomial T-S fuzzy system were studied in many papers such as [18], and [21]-[22]. A non-PDC control design for a polynomial T-S fuzzy system by using control Lyapunov function and Songtag’s formula was proposed in [18]. The observer-based controllers for the polynomial T-S fuzzy system with immeasurable premise variables were synthesized in [21] and [23]. In [22], the authors proposed a new approach for stability analysis and controller design for a general polynomial T-S fuzzy system in which the polynomial Lyapunov function candidate does not need to satisfy any constraint. Additionally, the controller synthesis for discrete time polynomial T-S fuzzy systems without and with delay time was developed in [19] and [20], respectively. From the above review, it becomes obvious that the polynomial T-S fuzzy system has been paid attention increasingly and it extends the study scope larger than the conventional T-S fuzzy system does for nonlinear control systems.

In a wide range of real-life and practical systems, all or some of state variables are immeasurable or difficult to obtain by using the measurement devices due to both technical and economic issues. These states, however, are really necessary for system supervision and controller design. Therefore, the observer design was taken notice increasingly. Regarding the observer synthesis for the polynomial T-S fuzzy systems, various methods have been presented in the past few years [25]-[28]. In [25], a synthesis of both controller and observer for polynomial T-S fuzzy system was proposed to guarantee the system stability and the state estimation simultaneously. In addition, the method for designing observer and controller simultaneously for both continuous and discrete time polynomial T-S fuzzy systems were presented in [27] and [28], respectively.
In practice, there are a large number of systems affected by many kinds of uncertainties. The presence of uncertainties in the system makes the design of the observer and controller for the system much more difficult. In our literature survey, a variety of studies dealing with uncertain problems of T-S fuzzy system were found in [29]-[33], however, there are only few papers concerning the polynomial T-S fuzzy system with uncertainties. Recently, the robust controller synthesis for the polynomial T-S fuzzy system with uncertainties was investigated in [20] and [34], in which the uncertainties in these papers must satisfy the norm-bounded constraints. Regarding the observer design for uncertain polynomial T-S fuzzy system, more recently, the papers [28] and [35] proposed the method to synthesize the controller and observer simultaneously. These papers not only eliminated the influence of uncertainties but also guaranteed the state estimation errors approaching to zero asymptotically. It should be noted that the uncertainties in the above papers must be under some bounds. If the upper bound of uncertainties is unknown, or only observer without controller is designed, the methods in [28], [35] will not work. To address this shortcoming, we propose in this paper a new approach based on the unknown input method to synthesize the observer for the uncertain polynomial T-S fuzzy system where the bounded constraints of uncertainties are not given and the controller is not needed to be designed simultaneously to eliminate the influence of uncertainties.

Recently, there have been several papers proposed the method to design observer based on the unknown input method [36]-[38] for T-S fuzzy system and unknown input polynomial T-S fuzzy system in [39] and [40]. Unfortunately, the results in [39] and [40] have two limitations. The first is it is difficult to find the feasible solution for the parameter matrices to design the observer. The details will be explained in Section II. The other is the proposed method using the common Lyapunov function candidate that often leads to a much conservative result. In order to overcome the above disadvantages, this paper proposes a new form of observer and uses a non-common Lyapunov function [41]-[42] to derive the conditions for the observer synthesis.

The paper is organized as follows. In Section II, we describe the considered polynomial fuzzy system model with uncertainties and point out the main problems to be resolved in this study. In Section III, the main theorems and observer synthesis procedures are proposed. In Section IV, two examples are presented to illustrate the effectiveness of the synthesized observer. Finally, a conclusion is presented in Section V.

**Notations:** $A > 0$ denote the positive definite matrix $A$; $A^T$ denotes the transpose of matrix $A$; $A^{-1}$ denotes the inverse of $A$; $A^+$ denotes the Moore-Penrose pseudo-inverse of $A$ and $A^+ = (A^T A)^{-1} A^T$. The symbol $9^{r \times m}$ denotes the set of $n \times m$ matrices; $I$ denotes the identity matrix; the asterisk (*) denotes the transposed elements of the symmetric matrix.

II. SYSTEM MODEL AND PROBLEM DESCRIPTION

A. System model

Let us consider the nonlinear system presented as follows

\[
\dot{x} = f(x,u,\theta) 
\]

\[
y(t) = Cx(t) 
\]

where $f$ is the nonlinear function including possibly uncertainties, $x$ is the state vector, $\theta$ is the vector of possibly time-varying parameter. The equation (1b) is a linear output equation with constant matrix $C$, and $y$ is the output vector. On the basis of the sector nonlinearity method [43], suppose the nonlinear system (1) can be represented as the class I polynomial T-S fuzzy system as follows [27]:

**Rule i :**

IF $\theta_i(t)$ is $Q_i$, ..., and $\theta_j(t)$ is $Q_j$, THEN

\[
\dot{x}(t) = (A_i(\xi(t)) + \Delta A_i(\xi(t)))x(t) 
\]

\[
+ (B_i(\xi(t)) + \Delta B_i(\xi(t)))u(t) 
\]

\[
y(t) = Cx(t), \quad i = 1, 2, ..., r, 
\]

where $\theta(t) = [\theta_1(t), \theta_2(t), ..., \theta_n(t)]$ is the vector of measurable premise variables and $Q_i$ ( $i=1,2,...,r$ ; $j=1,2,...,s$ ) is the fuzzy set, $r$ is the number of rules, and $s$ is the number of premise variables $\theta_i(t)$ . Suppose $x(t) \in \mathbb{R}^n$ is the unavailable state vector, $u(t) \in \mathbb{R}^n$ is the input vector and $y(t) \in \mathbb{R}^p$ is the output. It is noted that $\xi(t)$ is the measurable variable that could be a function of external variable, output $y(t)$, and/or time. The polynomial matrices $A_i(\xi(t)) \in \mathbb{R}^{r \times m}$, $B_i(\xi(t)) \in \mathbb{R}^{r \times m}$ are known polynomial matrices of the states and inputs, respectively. Moreover, $\Delta A_i(\xi(t))$ and $\Delta B_i(\xi(t))$ are the uncertainties of $A_i(\xi(t))$ and $B_i(\xi(t))$, respectively.

The overall uncertain polynomial T-S fuzzy system inferred from the plant rules of (2) is described below:

\[
\begin{cases}
\dot{x}(t) = \sum_{i=1}^{r} \beta_{i}(\theta(t))[ (A_i(\xi(t)) + \Delta A_i(\xi(t)))x(t) 
\]

\[
+ (B_i(\xi(t)) + \Delta B_i(\xi(t)))u(t) ] 
\]

\[
y(t) = Cx(t), 
\]

where $w_{i}(\theta(t)) = \prod_{j=1}^{r} Q_{ij}(\theta(t)), \quad \sum_{i=1}^{r} w_{i}(\theta(t)) > 0 \quad w_{i}(\theta(t)) \geq 0$ , $\beta_{i}(\theta(t)) \geq 0$, and $\sum_{i=1}^{r} \beta_{i}(\theta(t)) = 1$. 

\[
\beta_{i}(\theta(t)) = \frac{w_{i}(\theta(t))}{\sum_{i=1}^{r} w_{i}(\theta(t))} 
\]
Remark 1: As presented in [27], the polynomial T-S fuzzy system is classified into three types (class I, class II, class III). In this paper, we only consider the class I polynomial T-S fuzzy system which has the polynomial system matrices depend on the measurable time-varying variable $\zeta(t)$.

Remark 2: If $\zeta(t)$ is a constant, the system matrices $A_t(\zeta(t))$ and $B_t(\zeta(t))$ become constant matrices $A_t$ and $B_t$, the polynomial T-S fuzzy system (2) becomes the conventional T-S fuzzy system.

Remark 3: The uncertainties $\Delta A_t(\zeta(t))$ and $\Delta B_t(\zeta(t))$ depend on $\zeta(t)$, it means that the uncertainties are more practical compared to the uncertainties in [20], [28], and [34] which are only dependent on time. In addition, the bound conditions of these uncertainties are not given.

Remark 4: A nonlinear system can be transformed to a polynomial T-S fuzzy system of Class I by using the sector nonlinearity method, if the system matrix $A$ and input matrix $B$ of the polynomial T-S fuzzy system contain only the measurable variables that could be a function of external variable, output $y(t)$, and/or time. If some un-available state variables are inside $A$ and/or $B$, then these systems are considered as a polynomial T-S fuzzy system of Class II and III. Class II and III are more complicated to be studied which are not considered in this paper. The polynomial T-S fuzzy of class I, which is an extension of the general T-S fuzzy system, can significantly reduce the number of fuzzy rules for presenting the original nonlinear because the premise variable has been put inside the system matrices [27].

Proposition 1 [21]: If $p(x(t))$ is a SOS, then $p(x(t))$ can be rewritten as $p(x(t)) = \sum_{i=1}^{n} q_i(x(t))^3$, where $q_i(x(t))$ is a polynomial in $x(t)$. Therefore, when $p(x(t))$ is determined as the SOS, it implies that $p(x(t)) \geq 0$.

B. Problem description

Suppose that when all or some state variables $x(t)$ of the polynomial T-S fuzzy system (3) are unavailable, it is necessary to synthesize an observer to estimate these unavailable state variables. Hence, this paper aims to design an observer for the system (3) to guarantee the estimated state variables approaching to real states. As discussed before, [28] and [35] dealt with the problem of observer design for uncertain polynomial T-S fuzzy systems. However, in order to eliminate the influences of uncertainties and estimate state variables simultaneously, the methods in [28] and [35] designed an observer-based controller to achieve this objective. In addition, the upper bounds of uncertainties have to be given in advance; otherwise, it is infeasible to design the observer. In order to overcome these drawbacks, in this paper, we propose the new approach based on the unknown input method to synthesize an observer for the uncertain polynomial T-S fuzzy system (3).

If the unknown input observer form (observer (7) in [40]) is considered

$$
\begin{align*}
\dot{z}(t) &= \sum_{i=1}^{r} \beta_i(\theta(t))[N_i(\zeta(t))z(t) + G_i(\zeta(t))u(t) + L_i(\zeta(t))y(t)] \\
\dot{x}(t) &= z(t) - Ey(t), i = 1, 2, \ldots, r.
\end{align*}
$$

(4)

It is noted that the matrix $E$ in (4) is a constant matrix and it is hard to satisfy the condition $(P + SC)R(\gamma) = 0$ ((21) in Theorem 1 of [40]), where $S = PE$, since $P$, $S$, and $C$ are constant matrices while the matrix $R(\gamma)$ is the polynomial matrices. Due to the above analyses, this study tries to propose a new approach to synthesize a specific form of the observer for the uncertain polynomial T-S fuzzy system. Before the main derivation, the following two assumptions are needed.

Assumption 1: Assume the matching conditions $\Delta A_t(\zeta(t)) = D(\zeta(t))\Delta A_t(\zeta(t))$ and $\Delta B_t(\zeta(t)) = D(\zeta(t))\Delta B_t(\zeta(t))$ are satisfied, where $D(\zeta(t)) \in \mathbb{R}^{m \times n}$ is a full normal column rank matrix (see [45]-[47]), $\Delta A_t(\zeta(t)) \in \mathbb{R}^{m \times n}$ and $\Delta B_t(\zeta(t)) \in \mathbb{R}^{m \times m}$ are time-varying uncertain matrices which depend on $\zeta(t)$.

Assumption 2: The matrices $C$ and $D(\zeta(t))$ are full row and normal column ranks, respectively, and the normal rank of $(CD(\zeta(t)))$ is equal to the normal rank of $D(\zeta(t))$.

Remark 5: The Assumption 1 is necessary to transform the uncertainties to unknown inputs and the Assumption 2 is needed to guarantee the existence of general solutions of matrix equation which will appear in the proof of Theorem 2.

III. OBSERVER SYNTHESIS

Firstly, on the basis of the Assumption 1, the uncertain polynomial T-S fuzzy system is transformed to the unknown input polynomial T-S fuzzy system as follows.

Based on the Assumption 1, let us define $\Delta \hat{A}_t(\zeta(t))x(t) = \gamma_i(t)$, $\Delta \hat{B}_t(\zeta(t))u(t) = \sigma_i(t)$, and where $\gamma_i(t) \in \mathbb{R}^q$ and $\sigma_i(t) \in \mathbb{R}^q$ are considered the unknown inputs. Therefore, the system (3) can be rewritten as follows:

$$
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} \beta_i(\theta(t))[A_i(\zeta(t))x(t) + B_i(\zeta(t))u(t) + D(\zeta(t))\gamma_i(t)] \\
&\quad + D(\zeta(t))\sigma_i(t)] \\
y(t) &= Cx(t), i = 1, 2, \ldots, r,
\end{align*}
$$

(5)

Let $\omega_i(t) = \gamma_i(t) + \sigma_i(t)$ be substituted into (5), we obtain
\begin{equation}
\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta(t)) \left[ A_i(\xi(t)) x(t) + B_i(\xi(t)) u(t) + D(\xi(t)) \omega(t) \right] + y(t) = C x(t), \quad i = 1, 2, \ldots, r,
\end{equation}

(6)

Now, the uncertain polynomial T-S fuzzy system (3) has been transformed to unknown input polynomial T-S fuzzy system (6). Therefore, the observer will be synthesized to estimate unavailable states of the system (6) rather than of the system (3).

Re-consider the form of observer (4) but with a polynomial matrix \( E(\xi(t)) \) as follows

\begin{equation}
\dot{\hat{x}}(t) = \sum_{i=1}^{r} \beta_i(\theta(t)) \left[ N_i(\xi(t)) x(t) + G_i(\xi(t)) u(t) + L_i(\xi(t)) y(t) \right]
\end{equation}

(7a)

\begin{equation}
\hat{x}(t) = z(t) - E(\xi(t)) y(t)
\end{equation}

(7b)

The \( \hat{x}(t) \) is the estimation of the state variable \( x(t) \) and \( z(t) \) is the state vector of the observer.

Let us define the estimation error

\begin{equation}
e(t) = x(t) - \hat{x}(t)
\end{equation}

(8)

and substituting (7b) into (8) yields

\begin{equation}
e(t) = x(t) - z(t) + E(\xi(t)) y(t)
\end{equation}

(9)

where \( M(\xi(t)) = I + E(\xi(t)) C \).

Computing the derivative of estimation error \( e(t) \) in (9) yields

\begin{equation}
\dot{e}(t) = \frac{\partial M(\xi(t))}{\partial \xi(t)} \hat{x}(t) x(t) + M(\xi(t)) \hat{x}(t) - \hat{x}(t)
\end{equation}

(10)

From (10), it is clearly seen that the term \( \frac{\partial M(\xi(t))}{\partial \xi(t)} \hat{x}(t) x(t) \) is very complicated because of the terms of differentiation. Therefore, this paper proposes a new observer form as follows

\begin{equation}
\dot{\hat{x}}(t) = \sum_{i=1}^{r} \beta_i(\theta(t)) \left[ N_i(\xi(t)) \hat{x}(t) + G_i(\xi(t)) u(t) + L_i(\xi(t)) y(t) \right] + F(\xi(t)) \hat{y}(t)
\end{equation}

(11)

where \( N_i(\xi(t)) \in \mathbb{R}^{m_x} \), \( G_i(\xi(t)) \in \mathbb{R}^{m_u} \), \( L_i(\xi(t)) \in \mathbb{R}^{m_y} \), \( F(\xi(t)) \in \mathbb{R}^{m_x} \) are polynomial matrices to be determined later. \( \hat{x}(t) \) is the estimation of the real state variable \( x(t) \).

From (11), we have

\begin{equation}
\dot{\hat{y}}(t) = C \hat{x}(t)
\end{equation}

(12)

\begin{equation}
= \sum_{i=1}^{r} \beta_i(\theta(t)) \left[ C A_i(\xi(t)) x(t) + CB_i(\xi(t)) u(t) + CD(\xi(t)) \omega_i(t) \right]
\end{equation}

From (8), we have

\begin{equation}
\dot{\hat{x}}(t) = \hat{x}(t) - \dot{x}(t)
\end{equation}

(13)

Substituting (6) and (11) into (13) yields

\begin{equation}
\dot{e}(t) = \sum_{i=1}^{r} \beta_i(\theta(t)) \left[ A_i(\xi(t)) x(t) + B_i(\xi(t)) u(t) + D(\xi(t)) \omega(t) \right] + \sum_{i=1}^{r} \beta_i(\theta(t)) \left[ N_i(\xi(t)) \hat{x}(t) + G_i(\xi(t)) u(t) + L_i(\xi(t)) y(t) \right]
\end{equation}

(14)

\begin{equation}
+ L_i(\xi(t)) y(t) + F(\xi(t)) \hat{y}(t)
\end{equation}

Combining (12) and (14), the obtained result is

\begin{equation}
\dot{e}(t) = \sum_{i=1}^{r} \beta_i(\theta(t)) \left[ A_i(\xi(t)) x(t) + B_i(\xi(t)) u(t) + D(\xi(t)) \omega(t) \right] - N_i(\xi(t)) \dot{x}(t) - G_i(\xi(t)) \omega(t) - L_i(\xi(t)) y(t)
\end{equation}

\begin{equation}
- F(\xi(t)) [CA_i(\xi(t)) x(t) + CB_i(\xi(t)) u(t) + CD(\xi(t)) \omega_i(t)]
\end{equation}

\begin{equation}
= \sum_{i=1}^{r} \beta_i(\theta(t)) \left[ A_i(\xi(t)) x(t) - F(\xi(t)) CA_i(\xi(t)) x(t) - L_i(\xi(t)) C \right]
\end{equation}

\begin{equation}
- N_i(\xi(t)) \dot{x}(t) + B_i(\xi(t)) u(t) + F(\xi(t)) CB_i(\xi(t)) u(t) - G_i(\xi(t)) \omega(t) + D(\xi(t)) \omega_i(t) - F(\xi(t)) CD(\xi(t)) \omega_i(t)]
\end{equation}

(15)

In the following proof of the Theorem 1, an assumption is needed.

**Assumption 3** \([41]\): Assume that \( |\dot{\beta}_k(\theta(t))| \leq \dot{\epsilon}_k \), \( k = 1, 2, \ldots, r \), where \( \dot{\epsilon}_k > 0 \) is a constant.

**Theorem 1**: Under Assumptions 1, 2, and 3, the estimation error (8) with the observer (11) converges to zero asymptotically if there exist polynomial matrices \( F(\xi(t)) \), \( N_i(\xi(t)) \), \( L_i(\xi(t)) \), \( G_i(\xi(t)) \) and symmetric matrix \( P_i \) such that the following conditions hold with \( i = 1, 2, \ldots, r \).

\begin{equation}
A_i(\xi(t)) - F(\xi(t)) CA_i(\xi(t)) - L_i(\xi(t)) C - N_i(\xi(t)) = 0
\end{equation}

(16)

\begin{equation}
B_i(\xi(t)) - F(\xi(t)) CB_i(\xi(t)) - G_i(\xi(t)) = 0
\end{equation}

(17)

\begin{equation}
D(\xi(t)) - F(\xi(t)) CD(\xi(t)) = 0
\end{equation}

(18)

\begin{equation}
\psi_{i1}^T (P_i - \dot{\epsilon}_i I) \psi_{i1} \text{ is SOS}
\end{equation}

(19)

\begin{equation}
- \psi_{i2}^T \left( \sum_{k=1}^{r} \dot{\epsilon}_k^T P_k + N_i^T (\xi(t)) P_i + P_i N_i(\xi(t)) + \varepsilon_i (\xi(t)) I \right) \psi_{i2} \text{ is SOS}
\end{equation}

(20)

where \( \psi_{i1} \) and \( \psi_{i2} \) are vectors with appropriate dimensions that do not depend on \( \xi(t) \), \( \epsilon_i > 0 \), and \( \varepsilon_i(\xi(t)) > 0 \) at \( \xi(t) \neq 0 \).

**Proof**: If the conditions (16)-(18) of Theorem 1 hold, then (15) becomes
\[ \dot{e}(t) = \sum_{i=1}^{r} \beta_i(\theta) [N_i(\zeta(t))e(t)] \] (21)

Select the non-common Lyapunov function as follows

\[ V(t) = \sum_{i=1}^{r} \beta_i(\theta)e^T(t)P_i e(t) \] (22)

It is noted that if the condition (19) holds, it means that the symmetric \( P_i > 0 \).

Taking the derivative of Lyapunov function results in

\[ \dot{V}(t) = \sum_{i=1}^{r} \beta_i(\theta)e^T(t)P_i e(t) + \sum_{i=1}^{r} \beta_i(\theta)[e^T(t)\dot{P}_i e(t) + e^T(t)\dot{e}(t)] \].

Substituting (21) into (23), and on the basis of the Assumption 3 produces

\[ \dot{V}(t) \leq \sum_{i=1}^{r} \sum_{j=1}^{r} \beta_i(\theta) \mu_i(\zeta(t))P_i e(t) + \sum_{i=1}^{r} \sum_{j=1}^{r} \beta_i(\theta) \nu_i(\zeta(t))P_i e(t) \]

\[ + e^T(t)P_i N_i(\zeta(t))e(t) \leq \sum_{i=1}^{r} \sum_{j=1}^{r} \beta_i(\theta) \mu_i(\zeta(t))P_i e(t) + \sum_{i=1}^{r} \sum_{j=1}^{r} \beta_i(\theta) \nu_i(\zeta(t))P_i e(t) \]

\[ \times [N_i^T(\zeta(t))P_i + P_i N_i(\zeta(t))] \]

\[ \leq \sum_{i=1}^{r} \sum_{j=1}^{r} \beta_i(\theta) \mu_i(\zeta(t))P_i e(t) + \sum_{i=1}^{r} \sum_{j=1}^{r} \beta_i(\theta) \nu_i(\zeta(t))P_i e(t) \]

It can be seen from (24) that if the condition (20) holds, then \( \dot{V}(t) < 0 \), it means that the estimation error (8) approaches zero asymptotically. The proof is completed.

**Remark 5:** The term positive \( \varepsilon_1 \) and \( \varepsilon_2(\zeta(t)) > 0 \) are used in Theorem 1 to guarantee \( P_i \) is positive define matrix and \( \sum_{i=1}^{r} \beta_i(\theta) \mu_i(\zeta(t))P_i + \sum_{i=1}^{r} \beta_i(\theta) \nu_i(\zeta(t))P_i \) is negative define matrix instead of semi-positive and semi-negative define.

In order to synthesize the observer (11), all conditions (16)-(20) must be solved to determine the parameters \( N_i(\zeta(t)) \), \( G_i(\zeta(t)) \), \( L_i(\zeta(t)) \), and \( F(\zeta(t)) \) of the observer (11). However, the condition (20) is a polynomial BMI (Bilinear Matrix Inequality) which cannot be solved by using SOS TOOL in Matlab, therefore, we need to transform it into the polynomial Linear Matrix Inequalities (LMIs). The following Theorem 2 will be with polynomial LMI.

**Lemma 1** [44]: There are two matrices \( A \in \mathbb{R}^{m \times n} \) with \( m \geq n \), \( B \in \mathbb{R}^{k \times n} \) and suppose that \( BA^*A = B \). Then any matrix of the form \( X = BA^* + Y(I - AA^*) \) is a solution of \( XA = B \), where \( Y \in \mathbb{R}^{k \times n} \) is an arbitrary matrix and \( A^* \) is defined as \( A^* = (A^T A)^{-1}A^T \) which is the Moore-Penrose pseudo-inverse of \( A \).

**Lemma 2:** Let \( A, S, P, \) and \( R \) be matrices with proper sizes. The following two inequalities are equivalent:

i) \( R + A^T P + PA < 0 \)

ii) \( \exists S : \begin{bmatrix} A^T S^T + SA + R & P - S + A^T S^T \\ P - S^T + SA & -S - S^T \end{bmatrix} < 0 \)

**Proof of Lemma 2:**

* ii) implication i): We pre-multiply and post-multiply (ii) with \( \begin{bmatrix} I^T & A \end{bmatrix} \) and \( \begin{bmatrix} I \end{bmatrix} \) respectively which yields

\[ \begin{bmatrix} I^T & A \end{bmatrix} \begin{bmatrix} A^T S^T + SA + R & P - S + A^T S^T \\ P - S^T + SA & -S - S^T \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} = R + A^T P + PA < 0 \]

* i) implication ii): From i) we have

\[ (R + A^T P + PA) + (A^T S^T - A^T S^T) + (SA - SA) \]

\[ + (A^T S^T A - A^T S^T A) + (A^T SA - A^T SA) < 0 \]

\[ A^T S^T + SA + PA - SA + A^T S^T A + A^T P - A^T S^T + A^T SA - A^T SA < 0 \] (*)

We can rewrite (*) in the form of matrix as follows

\[ \begin{bmatrix} I^T & A \end{bmatrix} \begin{bmatrix} A^T S^T + SA + R & P - S + A^T S^T \\ P - S^T + SA & -S - S^T \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} < 0 \]

That leads to

\[ \begin{bmatrix} A^T S^T + SA + R & P - S + A^T S^T \\ P - S^T + SA & -S - S^T \end{bmatrix} < 0 \]

Therefore i) and ii) are equivalent. The proof is completed.

**Theorem 2:** Under Assumptions 1, 2, and 3, the estimation error (8) with the observer (11) converges to zero asymptotically if there exist polynomial matrices \( F(\zeta(t)) \), \( N_i(\zeta(t)) \), \( L_i(\zeta(t)) \), \( G_i(\zeta(t)) \), \( K(\zeta(t)) \), \( X(\zeta(t)) \), \( Q(\zeta(t)) \), and symmetric matrix \( P \) such that the following conditions hold for \( i = 1, 2, \ldots, r \).

\[ v_i^T(P - \varepsilon_i I)w_i \text{ is SOS} \]

\[ -v_i^T \begin{bmatrix} \Phi^{(1)}_i \\ \Omega^{(1)}_i \end{bmatrix} - X(\zeta(t)) - X^T(\zeta(t)) + \varepsilon_2(\zeta(t))I \]

\[ v_i \text{ is SOS} \]
\begin{equation}
\Phi_\alpha^{(1)} = \sum_{k=1}^{K} \hat{e}_k + (A_1(\zeta(t)) - U(\zeta(t))CA_1(\zeta(t)))^T X^T(\zeta(t))
\end{equation}

\begin{equation}
- (V(\zeta(t))CA(\zeta(t)))^T K^*(\zeta(t)) - C^T Q_i^*(\zeta(t) + X(\zeta(t))A_1(\zeta(t))
\end{equation}

\begin{equation}
- U(\zeta(t))CA(\zeta(t)) - K(\zeta(t))(V(\zeta(t))CA(\zeta(t))) - \bar{Q}(\zeta(t))C
\end{equation}

\begin{equation}
\Omega_y^{(1)} = P_j - X^T(\zeta(t)) + X(\zeta(t))A_1(\zeta(t)) - U(\zeta(t))CA(\zeta(t))
\end{equation}

\begin{equation}
- K(\zeta(t))(V(\zeta(t))CA(\zeta(t))) - \bar{Q}(\zeta(t))C
\end{equation}

(27)

where \( \nu_1 \) and \( \nu_2 \) are vectors with appropriate dimensions that do not depend on \( \zeta(t) \), \( \varepsilon_1 > 0 \), \( \varepsilon_2(\zeta(t)) > 0 \) at \( \zeta(t) \neq 0 \), and

\begin{equation}
(CD(\zeta(t)))^* = (((CD(\zeta(t)))^T CD(\zeta(t))))^{-1}(CD(\zeta(t)))^T
\end{equation}

\begin{equation}
U(\zeta(t)) = D(\zeta(t))(CD(\zeta(t)))^*
\end{equation}

\begin{equation}
V(\zeta(t)) = (I - (CD(\zeta(t)))(CD(\zeta(t)))^*)
\end{equation}

\begin{equation}
K(\zeta(t)) = X(\zeta(t))Y(\zeta(t))
\end{equation}

\begin{equation}
\bar{Q}(\zeta(t)) = X(\zeta(t))L_1(\zeta(t))
\end{equation}

Moreover, the parameters of observer (11) are computed as follows

\begin{equation}
F(\zeta(t)) = U(\zeta(t)) + Y(\zeta(t)V(\zeta(t))
\end{equation}

\begin{equation}
G_1(\zeta(t)) = B(\zeta(t)) - F(\zeta(t))CB(\zeta(t))
\end{equation}

\begin{equation}
L_1(\zeta(t)) = X^{-1}(\zeta(t))Q_1(\zeta(t))
\end{equation}

\begin{equation}
N_1(\zeta(t)) = A_1(\zeta(t)) - F(\zeta(t))CA_1(\zeta(t)) - L_1(\zeta(t))
\end{equation}

(36)

Proof:
From the condition (20) of Theorem 1, it can be inferred that

\begin{equation}
\sum_{k=1}^{K} \hat{e}_k + P_j + N_1(\zeta(t)) < 0.
\end{equation}

(37)

Employing the Lemma 2 for (37) with slack variable \( X(\zeta(t)) \), one obtains

\begin{equation}
\begin{bmatrix}
\Phi_\alpha^{(1)} \\
\Omega_y^{(1)} \\
\end{bmatrix}
\begin{bmatrix}
- X(\zeta(t)) - X^T(\zeta(t))
\end{bmatrix}
\end{equation}

\begin{equation}
< 0
\end{equation}

(38)

where

\begin{equation}
\Phi_\alpha^{(1)} = \sum_{k=1}^{K} \hat{e}_k + (A_1(\zeta(t)) - U(\zeta(t))CA_1(\zeta(t)))^T X^T(\zeta(t)) + X(\zeta(t))N_1(\zeta(t))
\end{equation}

\begin{equation}
\Omega_y^{(1)} = P_j - X^T(\zeta(t)) + X(\zeta(t))N_1(\zeta(t)).
\end{equation}

(39)

From (18), we have

\begin{equation}
F(\zeta(t))CD(\zeta(t)) = D(\zeta(t))
\end{equation}

(41)

On the basis of the Lemma 1, the general solution of (41) is

\begin{equation}
F(\zeta(t)) = D(\zeta(t))(CD(\zeta(t)))^*
\end{equation}

\begin{equation}
+ Y(\zeta(t))(I - (CD(\zeta(t)))(CD(\zeta(t)))^*)
\end{equation}

(42)

where

\begin{equation}
(CD(\zeta(t)))^* = (((CD(\zeta(t)))^T CD(\zeta(t))))^{-1}(CD(\zeta(t)))^T.
\end{equation}

(43)

It is noted that the existence of the general solution (42) is guaranteed if only if the matrices \( C \) and \( D(\zeta(t)) \) satisfy the Assumption 2.

Let’s denote

\begin{equation}
U(\zeta(t)) = D(\zeta(t))(CD(\zeta(t)))^*
\end{equation}

(43)

\begin{equation}
V(\zeta(t)) = (I - (CD(\zeta(t)))(CD(\zeta(t)))^*).
\end{equation}

(44)

Substituting (43) and (44) into (42) yields

\begin{equation}
F(\zeta(t)) = U(\zeta(t)) + Y(\zeta(t)V(\zeta(t)).
\end{equation}

(45)

From (16) and (45), one obtains

\begin{equation}
N_i(\zeta(t)) = A_i(\zeta(t)) - U(\zeta(t))CA_i(\zeta(t)) - L_i(\zeta(t))
\end{equation}

(46)

Substituting (46) into (39), the obtained result is

\begin{equation}
\Phi_{\alpha}^{(1)} = \sum_{k=1}^{K} \hat{e}_k + (A_1(\zeta(t)) - U(\zeta(t))CA_1(\zeta(t))) - (Y(\zeta(t))
\end{equation}

\begin{equation}
x(\zeta(t))CA_1(\zeta(t)) - L_1(\zeta(t))\bar{C}^T X^T(\zeta(t)) + X(\zeta(t))(A_1(\zeta(t))
\end{equation}

\begin{equation}
- U(\zeta(t))CA(\zeta(t)) - (Y(\zeta(t))V(\zeta(t))CA_1(\zeta(t)) - L_1(\zeta(t))
\end{equation}

\begin{equation}
= \sum_{k=1}^{K} \hat{e}_k + (A_1(\zeta(t)) - U(\zeta(t))CA_1(\zeta(t)))^T \bar{C}^T X^T(\zeta(t))
\end{equation}

\begin{equation}
- (V(\zeta(t))CA_1(\zeta(t)))^T \bar{C}^T X^T(\zeta(t)) - C^T L_1(\zeta(t))X^T(\zeta(t))
\end{equation}

\begin{equation}
+ X(\zeta(t))(A_1(\zeta(t)) - U(\zeta(t))CA_1(\zeta(t))
\end{equation}

\begin{equation}
- X(\zeta(t))Y(\zeta(t))(V(\zeta(t))CA_1(\zeta(t))) - X(\zeta(t))L_1(\zeta(t))C.
\end{equation}

(47)

Substituting (46) into (40), we have

\begin{equation}
\Omega_y^{(1)} = P_j - X^T(\zeta(t)) + X(\zeta(t))(A_1(\zeta(t)) - U(\zeta(t))CA_1(\zeta(t))
\end{equation}

\begin{equation}
- Y(\zeta(t))V(\zeta(t))CA_1(\zeta(t)) - L_1(\zeta(t))C
\end{equation}

\begin{equation}
= P_j - X^T(\zeta(t)) + X(\zeta(t))(A_1(\zeta(t)) - U(\zeta(t))CA_1(\zeta(t))
\end{equation}

\begin{equation}
- X(\zeta(t))Y(\zeta(t))(V(\zeta(t))CA_1(\zeta(t))) - X(\zeta(t))L_1(\zeta(t))C.
\end{equation}

(48)

Let us define

\begin{equation}
K(\zeta(t)) = X(\zeta(t))Y(\zeta(t))
\end{equation}

(49)

\begin{equation}
\bar{Q}(\zeta(t)) = X(\zeta(t))L_1(\zeta(t))
\end{equation}

(50)

Substituting (49) and (50) into (47) and (48), we obtain
Let the system (54) be transformed into a polynomial T-S fuzzy system where $\zeta(t)$ of polynomial T-S fuzzy system is the output $y(t)$. Then

$$
\begin{align*}
\dot{x}(t) & = \sum_{i=1}^{2} \beta_i(\theta(t))[A_i(y)x(t)+B_i(y)u(t)] \\
y(t) & = Cx(t)
\end{align*}
$$

(55)

We can see that

$$
[\dot{\beta}_1(\theta(t))]= \left[ \begin{array}{c}
-x_1 \\
\sin(x_1) \\
\end{array} \right],
[\dot{\beta}_2(\theta(t))]= \left[ \begin{array}{c}
\dot{x}_1 \\
\sin(x_1) \\
\end{array} \right],
$$

then we assume $\bar{\sigma}_1=\bar{\sigma}_2=0.4$.

Suppose the system (55) is influenced by the uncertainties and the bounded constraints of these uncertainties are unknown. The system (55) is expressed in the following form

$$
\begin{align*}
\dot{x}(t) & = \sum_{i=1}^{2} \beta_i(\theta(t))[A_i(y)+\Delta A_i(y)]x(t)+(B_i(y)+\Delta B_i(y))u(t) \\
y(t) & = Cx(t)
\end{align*}
$$

(56)

where

$$
\Delta A_i(y) = \begin{bmatrix}
\cos(2y)/y & -2\sin(2y+1)/y \\
0.25\cos(2y) & -0.5\sin(2y+1)
\end{bmatrix},
$$

$$
\Delta A_2(y) = \begin{bmatrix}
-\sin(2y)/y & -2.4\cos(2y)/y \\
-0.25\sin(2y) & -0.6\cos(2y)
\end{bmatrix},
$$

$$
\Delta B_1(y) = \begin{bmatrix}
2.2\cos(y^2)/y & \sin(y^2)/y \\
0.55\cos(y^2) & 0.25\sin(y^2)
\end{bmatrix},
\Delta B_2(y) = \begin{bmatrix}
\sin(y^2)/y & 0.25\sin(y^2)
\end{bmatrix}.
$$

Transforming system (56) into the unknown input system yields

$$
\begin{align*}
\dot{x}(t) & = \sum_{i=1}^{2} \beta_i(\theta(t))[A_i(y)x(t)+B_i(y)u(t)+D(y)\omega(t)] \\
y(t) & = Cx(t)
\end{align*}
$$

(57)

where

$$
D(y) = \begin{bmatrix}
2/y \\
0.5
\end{bmatrix}.
$$

Step 1: We can see that $C,D(y)$ are full row and column ranks, and normal rank($CD(\zeta(t))$) = normal rank($D(\zeta(t))$) = 1, thus the Assumption 2 is satisfied.
Step 2: From (29) and (30),

\[
U(y) = \begin{bmatrix} 1 \\ y/4 \end{bmatrix}, \quad V(y) = 0;
\]

Step 3: Resolve the constraint (25) and (26) of Theorem 2 using the SOS TOOL of Matlab. We can obtain the values below

\[
P_1 = \begin{bmatrix} 1.0397 & -0.24409 \times 10^{-5} \\ -0.24409 \times 10^{-5} & 0.13246 \times 10^{-8} \end{bmatrix},
\]

\[
P_2 = \begin{bmatrix} 1.0369 & -0.24294 \times 10^{-5} \\ -0.24294 \times 10^{-5} & 0.11553 \times 10^{-8} \end{bmatrix},
\]

\[
K(y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Y(y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

\[
X = \begin{bmatrix} 0.0013842 & 0.20861 \times 10^{-10} \\ -0.32898 \times 10^{-5} & 0.33362 \times 10^{-8} \end{bmatrix},
\]

\[
Q_1(y) = \begin{bmatrix} 0.2772 \times 10^{-5} y^2 - 0.6543 \times 10^{-8} y + 1.0351 & 0 \\ -0.5871 \times 10^{-10} y^2 - 0.2509 \times 10^{-7} y - 0.2135 \times 10^{-4} & -0.24409 \times 10^{-5} \\ -0.24409 \times 10^{-5} & 0.13246 \times 10^{-8} \end{bmatrix},
\]

\[
Q_2(y) = \begin{bmatrix} 0.27083 \times 10^{-8} y^2 + 0.78949 \times 10^{-9} y + 1.0377 & 0 \\ -0.16966 \times 10^{-9} y^2 + 0.39287 \times 10^{-6} y + 0.34 \times 10^{-5} \\ -0.24409 \times 10^{-5} & 0.13246 \times 10^{-8} \end{bmatrix},
\]

Step 4 and Step 5: The matrix \( F(\xi(t)) \) is obtained from (33). We obtain \( N_j(\xi(t)) \) from (36), \( L_j(\xi(t)) \) from (35) and \( G_j(\xi(t)) \) from (34).

\[
N_j(y) = \begin{bmatrix} -0.2003 \times 10^{-5} y^3 + 0.4613 \times 10^{-5} y & 0 \\ -0.1955 \times 10^{-5} y^2 + 0.120436 \times 10^{-3} y & -747.7854 \\ 0.0156 y^2 + 7.52 y - 736744.09 & -y^2 - y/4 \end{bmatrix},
\]

\[
N_j(y) = \begin{bmatrix} -0.1955 \times 10^{-5} y^2 + 0.120436 \times 10^{-3} y & 0 \\ -0.0527y^2 - 117.7585 y - 740255.5170 & -y^2 + y/4 \\ -749.6637 & -0.24409 \times 10^{-5} \\ -0.24409 \times 10^{-5} & 0.13246 \times 10^{-8} \end{bmatrix},
\]

\[
L_j(y) = \begin{bmatrix} 0.195579 \times 10^{-5} y^2 - 0.120435 \times 10^{-3} y + 749.663746 \\ 0.052783y^2 + 117.758539 y + 740256.517083 \\ -0.24409 \times 10^{-5} & 0.13246 \times 10^{-8} \end{bmatrix},
\]

\[
G_j(y) = \begin{bmatrix} 0 \\ -y^4/4 \end{bmatrix}, \quad G_2(y) = \begin{bmatrix} 0 \\ -y^4/4 \end{bmatrix}, \quad F(y) = \begin{bmatrix} 1 \\ y/4 \end{bmatrix}.
\]

Here, we used Simulink tools to carry out the simulation. The initial states of the system \( x(0) = [-1.5 \ -30] \) and the estimated states are \( \hat{x}(0) = [-0.5 \ -50] \). The input used for simulation is \( u(t) = 0.2 \sin(t) \).
Figures 1-3 show the simulation results of the numerical example, in which the system (56) is perturbed by the uncertainties $\Delta A_i(x_i(t)), \Delta B_i(y(t))$, and $\Delta B_j(y(t))$. Figures 1 and 2 depict the states $x_1$, $x_2$ and their estimated signals. The estimation error is illustrated in Fig. 3. It can be seen from these figures that the estimated states can approach real states asymptotically. Hence, the proposed method is successful in synthesizing observer for polynomial T-S fuzzy system with uncertainties.

**Remark 6:** It is noted that if we use general T-S fuzzy system to represent the nonlinear system (54), both $\cos(x_i)$ and $x_i$ must be linearized and the local bound range of $x_i$ must be given. Then there will be four fuzzy rules in the T-S fuzzy system. However, if let a polynomial T-S fuzzy system be present the original nonlinear system, $x_i$ will be put in the system matrices, therefore the bound range of $x_i$ does not need to know and the number of fuzzy rules will be reduced to two. It means that the polynomial T-S fuzzy system will represent the system (54) more exactly over global region.

**B. Example 2**

In this example, we consider a practical dynamic model of an Inverted Pendulum on a cart [21]. The system is depicted in the following figure.

![Inverted Pendulum on a cart](image)

The nonlinear equation of the Inverted Pendulum on a cart is expressed as follows

$$
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= \frac{g \sin(x_1(t)) - a m_p L x_2^2(t) \sin(x_1(t)) \cos(x_1(t))}{4L/3 - a m_p L \cos^2(x_1(t))} - \frac{a \cos(x_1(t))u}{4L/3 - a m_p L \cos^2(x_1(t))} \\
\end{align*}
$$

where $x = [x_1 \ x_2] = [\theta \ \dot{\theta}]$ is a vector of states, $\theta$ is the angle (in radians), $\dot{\theta}$ is the angular velocity. $m_p = 2kg$, $M_c = 8kg$, $2L = 1m$ are the mass of Pendulum, Cart and the length of the pendulum, respectively; $a = 1/(m_p + M_c)$. $u(t)$ is the control input force imposed on the cart, and $x_i(t) \in [-70\pi/180, 70\pi/180]$. In order to reduce computational burden, the term $\sin(x_1(t)) \approx s_1 x_1(t)$, $\tan(x_1(t)) \approx t_1 x_1(t)$, $s_1 = 0.8578$ and $t_1 = 1.5534$.

Applying nonlinear sector to linearize this system, the nonlinear system (58) is represented as the following polynomial T-S fuzzy system. It is noted that $y = Cx(t) = [0 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2$, therefore, the result of the nonlinear system can be expressed in the form of (3) with $\zeta(t)$ being the output $y(t)$.

$$
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{p} \beta_i(x_i(t))[A_i(y)x(t) + B_i(y)u(t)] \\
y &= Cx(t)
\end{align*}
$$

where

$$
\begin{align*}
A_1(y) &= \begin{bmatrix} 0 & 1 \\ a_1(y) & 0 \end{bmatrix}, & A_2(y) &= \begin{bmatrix} 0 & 1 \\ a_2(y) & 0 \end{bmatrix}, & B_1 &= \begin{bmatrix} 0 \\ -f_{\text{fmin}} a \end{bmatrix}, \\
B_2 &= \begin{bmatrix} 0 \\ -f_{\text{fmax}} a \end{bmatrix}, & C &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \\
f_{\text{fmin}} &= 0.5222, & f_{\text{fmax}} &= 1.7647 \\
a_1(y) &= f_{\text{fmin}}(gt_1 - am_p Ly^2 s_1), & a_2(y) &= f_{\text{fmax}}(gt_1 - am_p Ly^2 s_1)
\end{align*}
$$

The premise variables

$$
\beta_i(x_i(t)) = \frac{f_i(x_i(t)) - f_{\text{fmax}}}{f_{\text{fmin}} - f_{\text{fmax}}}, \quad \beta_i(x_i(t)) = 1 - \beta_i(x_i(t));
$$

$$
\begin{align*}
f_i(x_i) &= \frac{\cos(x_i)}{4L/3 - a m_p L \cos^2(x_i)}
\end{align*}
$$

Suppose the (59) are affect by the following uncertainties, then

$$
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{p} \beta_i(x_i(t))[\Delta A_i(y)x(t) + (\Delta B_i(y) + \Delta B_i(y))u(t)] \\
y &= Cx(t)
\end{align*}
$$

$$
\begin{align*}
\Delta A_i(y) &= \begin{bmatrix} 0.21y^2 \sin(2y + 1) & -0.12y^2 \cos(y) \\ 0.55\sin(2y + 1) & -0.6\cos(y) \end{bmatrix}, \\
\Delta B_i(y) &= \begin{bmatrix} -0.11y^2 \cos(2y) & 0.02y^2 \sin(y) \\ -0.55\cos(2y) & 0.1\sin(y) \end{bmatrix},
\end{align*}
$$
\[ \Delta B_1(y) = \begin{bmatrix} 0.21y^2 \cos(y^2) \\ 1.05 \cos(y^2) \end{bmatrix}, \quad \Delta B_2(y) = \begin{bmatrix} 0.21y^2 \sin(y^2) \\ 1.05 \sin(y^2) \end{bmatrix}. \]

Transform (60) into unknown input system with 
\[ D(y) = \begin{bmatrix} 0.1y^2 \\ 0.5 \end{bmatrix}. \]
With \( x_i(t) \in \left[ -\frac{70\pi}{180}, \frac{70\pi}{180} \right] \), in this paper, we assume \( \delta_1 = \delta_2 = 2.5. \)

**Step 1:** It is seen that the matrices \( C, D(y(t)) \) are full row and column ranks, respectively, and normal rank\((CD(\zeta(t))) = \text{normal rank}(D(\zeta(t))) = 1 \), therefore, the Assumption 2 is satisfied.

**Step 2:** The matrices \( U_i(y) \) and \( V_i(y) \) are calculated:
\[ U(y) = \begin{bmatrix} y^2 / 5 \\ 1 \end{bmatrix}, \quad V(y) = 0. \]

**Step 3:** The matrices \( P_1, Q_i(y(t)), K(y(t)), Y(y(t)), \) and \( X(y(t)) \) are obtained as follows
\[ P_1 = \begin{bmatrix} 0.40015 \times 10^{-8} & 0.19773 \times 10^{-10} \\ 0.19773 \times 10^{-10} & 0.20074 \times 10^{-8} \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.40092 \times 10^{-8} & 0.07057 \times 10^{-10} \\ 0.07057 \times 10^{-10} & 0.34511 \times 10^{-8} \end{bmatrix}, \quad Y(y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad K(y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad X(y) = \begin{bmatrix} 0.250 \times 10^{-7} & 0.15309 \times 10^{-4} \\ 0.2297 \times 10^{-8} & 0.43039 \end{bmatrix}. \]
\[ Q_i(y) = \begin{bmatrix} -0.8162 \times 10^{-10}y^2 - 0.232 \times 10^{-11}y + 0.153 \times 10^{-4} \\ 0.69247 \times 10^{-4}y^2 - 0.40629 \times 10^{-10}y + 0.43036 \end{bmatrix}, \quad Q_2(y) = \begin{bmatrix} 0.883 \times 10^{-9}y^2 + 0.985 \times 10^{-16}y + 0.153 \times 10^{-4} \\ 0.702 \times 10^{-9}y^2 - 0.881 \times 10^{-10}y + 0.430 \end{bmatrix}. \]

**Step 4 and Step 5:** The parameters \( N_i(y), G_i(y), F_i(y), \) and \( L_i(y(t)) \) are computed.

The matrices \( N_1(y) \) and \( N_2(y) \) are expressed in (61) and (62), respectively,
\[ F_i(y) = \begin{bmatrix} y^2 / 5 \\ 1 \end{bmatrix}, \quad G_i(y) = \begin{bmatrix} (261y^2) / 25000 \\ 0 \end{bmatrix}. \]

\[ N_1(y) = \begin{bmatrix} (y^2((0.4447y^2) - 7.9496))/5 \\ 0 \end{bmatrix}, \quad G_1(y) = \begin{bmatrix} 0.0893y^2 + 0.81 \times 10^{-4}y - 0.0551 \\ -0.161 \times 10^{-3}y^2 + 0.9397 \times 10^{-10}y - 0.999 \end{bmatrix} \]
\[ N_2(y) = \begin{bmatrix} (y^2((0.1513y^2) - 26.8645))/5 \\ 0 \end{bmatrix}, \quad G_2(y) = \begin{bmatrix} 0.056y^2 - 0.4559 \times 10^{-8}y - 0.0325 \\ -0.163 \times 10^{-3}y^2 + 0.2048 \times 10^{-11}y - 0.9999 \end{bmatrix}. \]

The simulation results of the observer synthesis for the practical dynamic system of Inverted Pendulum on a Cart which is affected by uncertainties are illustrated in Figs. 5-7. Obviously, Fig. 5 and Fig. 6 show that estimation states \( \hat{x}_i \) and \( \hat{x}_i(t) \) respectively.
\( \hat{x}_i \), approach asymptotically to real states \( x_1 \) and \( x_2 \), respectively. In Fig. 7, it is clearly seen that the estimation errors \( e_1 \) and \( e_2 \) converge to zero asymptotically.

**Remark 7:** \( |\hat{\beta}_k(\theta(t))| \leq \vartheta_k, \ k = 1, 2, \ldots, r \), in Assumption 3 is used in the above two examples. However, in this study, the bound range of some or all state variables are not known, and the bound of the uncertainties is not known either; hence, the upper bounds \( \vartheta_1 \) and \( \vartheta_2 \) are difficult to estimate. Therefore, in this study, the values of \( \vartheta_1 \) and \( \vartheta_2 \) in Example 1 and 2 are selected by trial and error. If the chosen \( \vartheta_1 \) and \( \vartheta_2 \) make the observer synthesis be feasible and the simulation is successful, then the synthesized observer is effective. If the chosen \( \vartheta_1 \) and \( \vartheta_2 \) either make the observer synthesis be not feasible or the simulation is not successful, we need to choose another couple \( \vartheta_1 \) and \( \vartheta_2 \). Based on our experience, if the feasible observer is not obtained, we may decrease \( \vartheta_1 \) and/or \( \vartheta_2 \) to design it again; however, if the observer synthesis is feasible, but the simulation is not successful, we may increase \( \vartheta_1 \) and/or \( \vartheta_2 \). We have to admit that trial and error method is not reliable for the observer synthesis, it should be resolved in the future work.

**Remark 8:** It is worth noting that we assume that the upper bounds of the uncertainties in Example 1 and Example 2 are unavailable, therefore, the method in [21] and [28] will fail to design observer for the system (56) and (60). By using the proposed method with a new form of the observer (12), the observer has been synthesized successfully for the uncertain polynomial T-S fuzzy system, in which the bounded constraints are unknown and the controller did not need to be designed together with an observer for the purpose of eliminating the effects of uncertainties.

### V. Conclusion

This paper has proposed a new approach based on the unknown input method to synthesize the observer for the uncertain polynomial T-S fuzzy system. On the basis of non-common Lyapunov function and SOS technique, two main theorems which contain the conditions for observer synthesis have been derived. These conditions are solved easily by using SOS TOOL of Matlab. Finally, two examples have demonstrated to present the effectiveness of the proposed method. The main characteristics of this paper are as follows.

(i) There are no upper bound limits for uncertainties. (ii) The observer synthesis can be completed independently without designing controller together. (iii) The observer form is new and is notably completely different from other existing traditional form of observers.

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