Unknown Input Based Observer Synthesis for a Polynomial T-S Fuzzy Model System with Uncertainties

Van-Phong Vu, Wen-June Wang, *Fellow, IEEE*, Hsiang-Chieh Chen, and Jacek. M. Zurada, *Life Fellow, IEEE*

***Abstract*— This paper proposes a new approach based on the unknown input method to synthesize the observer for polynomial Takagi-Sugeno (T-S) fuzzy system with uncertainties. In this paper, the upper bounds of uncertainties are not given and the effect of uncertainties is eliminated without designing an extra controller. With the aids of the non-common Lyapunov theory and Matlab's tools of the Sum-of-Square (SOS), a new observer is synthesized in which the observer form is completely different from the traditional observer forms reported in previous papers. The conditions for the observer synthesis are much relaxed and the complexity of the design process is reduced. Finally, two illustrative examples are presented to demonstrate the effectiveness of the proposed method.**

***Index Terms*—Uncertain polynomial T-S fuzzy systems, observer synthesis, unknown inputs, Sum of Square (SOS).**

I. Introduction

Tagaki-Sugeno (T-S) fuzzy model [1]-[4] has received a great deal of attention in control system research area. This model has provided another way to solve the problems of nonlinear control systems. Therefore, many theories and control design methods for linear control systems can be applied in the T-S fuzzy system. In addition, a new approach for designing an optimal coordination controller based on the adaptive fuzzy dynamic programming and game theory for solving the consensus problem of multi-agent differential games was studied in [5]. There was a book [6] to study the Type-2 fuzzy logic in detail and a related study to design an adaptive slide mode controller for the interval Type-2 fuzzy systems was reported in [7]. In 2009, a more general form of the T-S fuzzy model called the polynomial T-S fuzzy model has been introduced in [8] and an interval Type 2 polynomial fuzzy model was investigated in [9]. This model allows the system matrices containing polynomial forms in its entries instead of only constant forms. With the supporting Sum-Of-Square (SOS) Tools in Matlab [10], the polynomial T-S fuzzy system can be considered as an effective tool for modeling nonlinear control systems. Recently, a large number of studies focused on the polynomial T-S fuzzy systems such as controller design, observer design, and stability analysis [8]-[24]. For example, the stability analyses for the polynomial T-S fuzzy systems by employing the multiple Lyapunov function and switching Lyapunov function were investigated in [13] and [14], respectively. Besides, the controller design and observer-based controller design for the polynomial T-S fuzzy system were studied in many papers such as [18], and [21] -[22]. A non-PDC control design for a polynomial T-S fuzzy system by using control Lyapunov function and Songtag's formula was proposed in [18]. The observer-based controllers for the polynomial T-S fuzzy system with immeasurable premise variables were synthesized in [21] and [23]. In [22], the authors proposed a new approach for stability analysis and controller design for a general polynomial T-S fuzzy system in which the polynomial Lyapunov function candidate does not need to satisfy any constraint. Additionally, the controller synthesis for discrete time polynomial T-S fuzzy systems without and with delay time was developed in [19] and [20], respectively. From the above review, it becomes obvious that the polynomial T-S fuzzy system has been paid attention increasingly and it extends the study scope larger than the conventional T-S fuzzy system does for nonlinear control systems.

In a wide range of real-life and practical systems, all or some of state variables are immeasurable or difficult to obtain by using the measurement devices due to both technical and economic issues. These states, however, are really necessary for system supervision and controller design. Therefore, the observer design was taken notice increasingly. Regarding the observer synthesis for the polynomial T-S fuzzy systems, various methods have been presented in the past few years [25] -[28]. In [25], a synthesis of both controller and observer for polynomial T-S fuzzy system was proposed to guarantee the system stability and the state estimation simultaneously. In addition, the method for designing observer and controller simultaneously for both continuous and discrete time polynomial T-S fuzzy systems were presented in [27] and [28], respectively.

Manuscript received October 19, 2016; revised February 9, 2017; accepted June 27, 2017. (*Corresponding author: Wen-June Wang)*

V. P Vu and W. J. Wang are with the Department of Electrical Engineering, National Central University, Jhong-Li 32001, Taiwan, R.O.C (e-mail: phongvv@hcmute.edu.vn ;wjwang@ee.ncu.edu.tw).

H. C. Chen is with the Department of Electrical Engineering, National United University, Miaoli 36063, Taiwan, R.O.C.(e-mail: chc@nuu.edu.tw).

J. M. Zurada is with the Department of Electrical and Computer Engineering , University of Louisville, Lousville, KY40292 USA, (e-mail: jacek.zurada@louisville.edu).

In practice, there are a large number of systems affected by many kinds of uncertainties. The presence of uncertainties in the system makes the design of the observer and controller for the system much more difficult. In our literature survey, a variety of studies dealing with uncertain problems of T-S fuzzy system were found in [29]-[33], however, there are only few papers concerning the polynomial T-S fuzzy system with uncertainties. Recently, the robust controller synthesis for the polynomial T-S fuzzy system with uncertainties was investigated in [20] and [34], in which the uncertainties in these papers must satisfy the norm-bounded constraints. Regarding the observer design for uncertain polynomial T-S fuzzy system, more recently, the papers [28] and [35] proposed the method to synthesize the controller and observer simultaneously. These papers not only eliminated the influence of uncertainties but also guaranteed the state estimation errors approaching to zero asymptotically. It should be noted that the uncertainties in the above papers must be under some bounds. If the upper bound of uncertainties is unknown, or only observer without controller is designed, the methods in [28], [35] will not work. To address this shortcoming, we propose in this paper a new approach based on the unknown input method to synthesize the observer for the uncertain polynomial T-S fuzzy system where the bounded constraints of uncertainties are not given and the controller is not needed to be designed simultaneously to eliminate the influence of uncertainties.

Recently, there have been several papers proposed the method to design observer based on the unknown input method [36]-[38] for T-S fuzzy system and unknown input polynomial T-S fuzzy system in [39] and [40]. Unfortunately, the results in [39] and [40] have two limitations. The first is it is difficult to find the feasible solution for the parameter matrices to design the observer. The details will be explained in Section II. The other is the proposed method using the common Lyapunov function candidate that often leads to a much conservative result. In order to overcome the above disadvantages, this paper proposes a new form of observer and uses a non-common Lyapunov function [41]-[42] to derive the conditions for the observer synthesis.

The paper is organized as follows. In Section II, we describe the considered polynomial fuzzy system model with uncertainties and point out the main problems to be resolved in this study. In Section III, the main theorems and observer synthesis procedures are proposed. In Section IV, two examples are presented to illustrate the effectiveness of the synthesized observer. Finally, a conclusion is presented in Section V.

Notations: $A > 0$ denote the positive definite matrix A; A^T denotes the transpose of matrix A; A^{-1} denotes the inverse of A; A^+ denotes the Moore-Penrose pseudo-inverse of A and $A^+ = (A^T A)^{-1} A^T$. The symbol $\Re^{n\times m}$ denotes the set of $n \times m$ matrices; I denotes the identity matrix; the asterisk (*) denotes the transposed elements of the symmetric matrix.

II. System Model and Problem Description

A. System model

Let us consider the nonlinear system presented as follows

$$\dot{x} = f(x, u, \theta) \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where f is the nonlinear function including possibly uncertainties, x is the state vector, θ is the vector of possibly time-varying parameter. The equation (1b) is a linear output equation with constant matrix C, and y is the output vector. On the basis of the sector nonlinearity method [43], suppose the nonlinear system (1) can be represented as the class I polynomial T-S fuzzy system as follows [27]:

Rule i:
IF $\theta_1(t)$ is Q_{i1}, …, and $\theta_s(t)$ is Q_{is}, THEN

$$\begin{aligned}\dot{x}(t) = {} & (A_i(\varsigma(t)) + \Delta A_i(\varsigma(t)))x(t) \\ & + (B_i(\varsigma(t)) + \Delta B_i(\varsigma(t)))u(t)\end{aligned} \tag{2a}$$

$$y(t) = Cx(t),\ i = 1,\ 2, \ldots,\ r, \tag{2b}$$

where $\theta(t) = [\theta_1(t),\ \theta_2(t), \ldots,\ \theta_s(t)]$ is the vector of measurable premise variables and Q_{ij} (i=1, 2, ..., r ; j= 1, 2, ..., s) is the fuzzy set, r is the number of rules, and s is the number of premise variables $\theta_i(t)$. Suppose $x(t) \in \Re^n$ is the unavailable state vector, $u(t) \in \Re^m$ is the input vector and $y(t) \in \Re^p$ is the output. It is noted that $\varsigma(t)$ is the measurable variable that could be a function of external variable, output $y(t)$, and/or time. The polynomial matrices $A_i(\varsigma(t)) \in \Re^{n\times n}$, $B_i(\varsigma(t)) \in \Re^{n\times m}$ are known polynomial matrices of the states and inputs, respectively. Moreover, $\Delta A_i(\varsigma(t))$ and $\Delta B_i(\varsigma(t))$ are the uncertainties of $A_i(\varsigma(t))$ and $B_i(\varsigma(t))$, respectively. The overall uncertain polynomial T-S fuzzy system inferred from the plant rules of (2) is described below:

$$\begin{cases}\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta(t))\big[(A_i(\varsigma(t)) + \Delta A_i(\varsigma(t)))x(t) \\ \qquad\qquad + (B_i(\varsigma(t)) + \Delta B_i(\varsigma(t)))u(t)\big] \\ y(t) = Cx(t),\end{cases} \tag{3}$$

where $w_i(\theta(t)) = \prod_{j=1}^{s} Q_{ij}(\theta(t))$, $\sum_{i=1}^{n} w_i(\theta(t)) > 0$, $w_i(\theta(t)) \ge 0$, $\beta_i(\theta(t)) = \dfrac{w_i(\theta(t))}{\sum_{i=1}^{r} w_i(\theta(t))} \ge 0$, and $\sum_{i=1}^{r} \beta_i(\theta(t)) = 1$.

Remark 1: As presented in [27], the polynomial T-S fuzzy system is classified into three types (class I, class II, class III). In this paper, we only consider the class I polynomial T-S fuzzy system which has the polynomial system matrices depend on the measurable time-varying variable $\varsigma(t)$.

Remark 2: If $\varsigma(t)$ is a constant, the system matrices $A_i(\varsigma(t))$ and $B_i(\varsigma(t))$ become constant matrices A_i and B_i, the polynomial T-S fuzzy system (2) becomes the conventional T-S fuzzy system.

Remark 3: The uncertainties $\Delta A_i(\varsigma(t))$ and $\Delta B_i(\varsigma(t))$ depend on $\varsigma(t)$, it means that the uncertainties are more practical compared to the uncertainties in [20], [28], and [34] which are only dependent on time. In addition, the bound conditions of these uncertainties are not given.

Remark 4: A nonlinear system can be transformed to a polynomial T-S fuzzy system of Class I by using the sector nonlinearity method, if the system matrix A_i and input matrix B_i of the polynomial T-S fuzzy system contain only the measurable variables that could be a function of external variable, output $y(t)$, and/or time. If some un-available state variables are inside A_i and/or B_i, then these systems are considered as a polynomial T-S fuzzy system of Class II and III. Class II and III are more complicated to be studied which are not considered in this paper. The polynomial T-S fuzzy of class I, which is an extension of the general T-S fuzzy system, can significantly reduce the number of fuzzy rules for presenting the original nonlinear because the premise variable has been put inside the system matrices [27].

Proposition 1 [21]: If $p(x(t))$ is a SOS, then $p(x(t))$ can be rewritten as $p(x(t))=\sum_{l=1}^{n} q_l(x(t))^2$, where $q_l(x(t))$ is a polynomial in $x(t)$. Therefore, when $p(x(t))$ is determined as the SOS, it implies that $p(x(t))\geq 0$.

B. Problem description

Suppose that when all or some state variables $x(t)$ of the polynomial T-S fuzzy system (3) are unavailable, it is necessary to synthesize an observer to estimate these unavailable state variables. Hence, this paper aims to design an observer for the system (3) to guarantee the estimated state variables approaching to real states. As discussed before, [28] and [35] dealt with the problem of observer design for uncertain polynomial T-S fuzzy systems. However, in order to eliminate the influences of uncertainties and estimate state variables simultaneously, the methods in [28] and [35] designed an observer-based controller to achieve this objective. In addition, the upper bounds of uncertainties have to be given in advance; otherwise, it is infeasible to design the observer. In order to overcome these drawbacks, in this paper, we propose the new approach based on the unknown input method to synthesize an observer for the uncertain polynomial T-S fuzzy system (3).

If the unknown input observer form (observer (7) in [40]) is considered

$$\begin{cases}\dot{z}(t)=\sum_{i=1}^{r}\beta_i(\theta(t))\left[N_i(\varsigma(t))z(t)+G_i(\varsigma(t))u(t)+L_i(\varsigma(t))y(t)\right]\\ \hat{x}(t)=z(t)-Ey(t), i=1,2\ldots r.\end{cases}\tag{4}$$

It is noted that the matrix E in (4) is a constant matrix and it is hard to satisfy the condition $(P+SC)R_i(y)=0$ ((21) in Theorem 1 of [40]), where S=PE, since P, S, and C are constant matrices while the matrix $R_i(y)$ is the polynomial matrices. Due to the above analyses, this study tries to propose a new approach to synthesize a specific form of the observer for the uncertain polynomial T-S fuzzy system. Before the main derivation, the following two assumptions are needed.

Assumption 1: Assume the matching conditions $\Delta A_i(\varsigma(t))=D(\varsigma(t))\Delta\tilde{A}_i(\varsigma(t))$ and $\Delta B_i(\varsigma(t))=D(\varsigma(t))\Delta\tilde{B}_i(\varsigma(t))$ are satisfied, where $D(\varsigma(t))\in\Re^{n\times q}$ is a full normal column rank matrix (see [45]-[47]), $\Delta\tilde{A}_i(\varsigma(t))\in\Re^{q\times n}$ and $\Delta\tilde{B}_i(\varsigma(t))\in\Re^{q\times m}$ are time-varying uncertain matrices which depend on $\varsigma(t)$.

Assumption 2: The matrices C and $D(\varsigma(t))$ are full row and normal column ranks, respectively, and the normal rank of $(CD(\varsigma(t)))$ is equal to the normal rank of $D(\varsigma(t))$.

Remark 5: The Assumption 1 is necessary to transform the uncertainties to unknown inputs and the Assumption 2 is needed to guarantee the existence of general solutions of matrix equation which will appear in the proof of Theorem 2.

III. Observer Synthesis

Firstly, on the basis of the Assumption 1, the uncertain polynomial T-S fuzzy system is transformed to the unknown input polynomial T-S fuzzy system as follows.

Based on the Assumption 1, let us define $\Delta\tilde{A}_i(\varsigma(t))x(t)=\gamma_i(t)$, $\Delta\tilde{B}_i(\varsigma(t))u(t)=\sigma_i(t)$, and where $\gamma_i(t)\in\Re^q$ and $\sigma_i(t)\in\Re^q$ are considered the unknown inputs. Therefore, the system (3) can be rewritten as follows:

$$\begin{cases}\dot{x}(t)=\sum_{i=1}^{r}\beta_i(\theta(t))\left[A_i(\varsigma(t))x(t)+B_i(\varsigma(t))u(t)+D(\varsigma(t))\gamma_i(t)\right.\\ \qquad\left.+D(\varsigma(t))\sigma_i(t)\right]\\ y(t)=Cx(t),\ i=1,\ 2,\ \ldots r,\end{cases}\tag{5}$$

Let $\omega_i(t)=\gamma_i(t)+\sigma_i(t)$ be substituted into (5), we obtain

$$\begin{cases}\dot{x}(t)=\sum_{i=1}^{r}\beta_i(\theta(t))\left[A_i(\varsigma(t))x(t)+B_i(\varsigma(t))u(t)+D(\varsigma(t))\omega_i(t)\right]\\ y(t)=Cx(t),\ i=1,\ 2,\ \ldots r,\end{cases} \quad (6)$$

Now, the uncertain polynomial T-S fuzzy system (3) has been transformed to unknown input polynomial T-S fuzzy system (6). Therefore, the observer will be synthesized to estimate unavailable states of the system (6) rather than of the system (3).

Re-consider the form of observer (4) but with a polynomial matrix $E(\varsigma(t))$ as follows

$$\begin{cases}\dot{z}=\sum_{i=1}^{r}\beta_i(\theta(t))[N_i(\varsigma(t))x(t)+G_i(\varsigma(t))u(t)+L_i(\varsigma(t))y(t)] & \text{(7a)}\\ \hat{x}(t)=z(t)-E(\varsigma(t))y(t) & \text{(7b)}\end{cases}$$

The $\hat{x}(t)$ is the estimation of the state variable $x(t)$ and $z(t)$ is the state vector of the observer.

Let us define the estimation error

$$e(t)=x(t)-\hat{x}(t) \quad (8)$$

and substituting (7b) into (8) yields

$$\begin{aligned}e(t)&=x(t)-z(t)+E(\varsigma(t))y(t)\\&=(I+E(\varsigma(t))C)x(t)-z(t)=M(\varsigma(t))x(t)-z(t)\end{aligned} \quad (9)$$

where $M(\varsigma(t))=I+E(\varsigma(t))C.$

Computing the derivative of estimation error $e(t)$ in (9) yields

$$\dot{e}(t)=\frac{\partial M(\varsigma(t))}{\partial\varsigma(t)}\dot{\varsigma}(t)x(t)+M(\varsigma(t))\dot{x}(t)-\dot{z} \quad (10)$$

From (10), it is clearly seen that the term $\frac{\partial M(\varsigma(t))}{\partial\varsigma(t)}\dot{\varsigma}(t)x(t)$ is very complicated because of the terms of differentiation. Therefore, this paper proposes a new observer form as follows

$$\begin{aligned}\dot{\hat{x}}=\sum_{i=1}^{r}\beta_i(\theta(t))\big[&N_i(\varsigma(t))\hat{x}(t)+G_i(\varsigma(t))u(t)\\&+L_i(\varsigma(t))y(t)+F(\varsigma(t))\dot{y}(t)\big]\end{aligned} \quad (11)$$

where $N_i(\varsigma(t))\in\Re^{n\times n}$, $G_i(\varsigma(t))\in\Re^{n\times m}$, $L_i(\varsigma(t))\in\Re^{n\times p}$, $F(\varsigma(t))\in\Re^{n\times p}$ are polynomial matrices to be determined later. $\hat{x}(t)$ is the estimation of the real state variable $x(t)$.

From (6), we have

$$\begin{aligned}\dot{y}(t)&=C\dot{x}(t)\\&=\sum_{i=1}^{r}\beta_i(\theta(t))[CA_i(\varsigma(t))x(t)+CB_i(\varsigma(t))u(t)+CD(\varsigma(t))\omega_i(t)]\end{aligned} \quad (12)$$

From (8), we have

$$\dot{e}(t)=\dot{x}(t)-\dot{\hat{x}}(t) \quad (13)$$

Substituting (6) and (11) into (13) yields

$$\begin{aligned}\dot{e}(t)&=\sum_{i=1}^{r}\beta_i(\theta(t))\big[A_i(\varsigma(t))x(t)+B_i(\varsigma(t))u(t)\\&+D(\varsigma(t))\omega_i(t)\big]-\sum_{i=1}^{r}\beta_i(\theta(t))\big[N_i(\varsigma(t))\hat{x}(t)+G_i(\varsigma(t))u(t)\\&+L_i(\varsigma(t))y(t)+F(\varsigma(t))\dot{y}(t)\big]\end{aligned} \quad (14)$$

Combining (12) and (14), the obtained result is

$$\begin{aligned}\dot{e}(t)&=\sum_{i=1}^{r}\beta_i(\theta(t))\big\{A_i(\varsigma(t))x(t)+B_i(\varsigma(t))u(t)+D(\varsigma(t))\omega_i(t)\\&-N_i(\varsigma(t))\hat{x}(t)-G_i(\varsigma(t))u(t)-L_i(\varsigma(t))y(t)\\&-F(\varsigma(t))[CA_i(\varsigma(t))x(t)+CB_i(\varsigma(t))u(t)+CD(\varsigma(t))\omega_i(t)]\big\}\\&=\sum_{i=1}^{r}\beta_i(\theta(t))\big[A_i(\varsigma(t))x(t)-F(\varsigma(t))CA_i(\varsigma(t))x(t)-L_i(\varsigma(t))\\&\times Cx(t)-N_i(\varsigma(t))\hat{x}(t)+B_i(\varsigma(t))u(t)-F(\varsigma(t))CB_i(\varsigma(t))u(t)\\&-G_i(\varsigma(t))u(t)+D(\varsigma(t))\omega_i(t)-F(\varsigma(t))CD(\varsigma(t))\omega_i(t)\big]\\&=\sum_{i=1}^{r}\beta_i(\theta(t))\big[(A_i(\varsigma(t))-F(\varsigma(t))CA_i(\varsigma(t))-L_i(\varsigma(t))C\\&-N_i(\varsigma(t)))x(t)+N_i(\varsigma(t))e(t)+(B_i(\varsigma(t))-F(\varsigma(t))CB_i(\varsigma(t))\\&-G_i(\varsigma(t)))u(t)+(D(\varsigma(t))-F(\varsigma(t))CD(\varsigma(t)))\omega_i(t)\big]\end{aligned} \quad (15)$$

In the following proof of the Theorem 1, an assumption is needed.

Assumption 3 [41]: Assume that $\left|\dot{\beta}_k(\theta(t))\right|\le\partial_k$, $k=1,\ 2,\ \ldots,\ r,$ where $\partial_k>0$ is a constant.

Theorem 1: Under Assumptions 1, 2, and 3, the estimation error (8) with the observer (11) converges to zero asymptotically if there exist polynomial matrices $F(\varsigma(t))$, $N_i(\varsigma(t))$, $L_i(\varsigma(t))$, $G_i(\varsigma(t))$ and symmetric matrix P_i such that the following conditions hold with i=1, 2, …, r.

$$A_i(\varsigma(t))-F(\varsigma(t))CA_i(\varsigma(t))-L_i(\varsigma(t))C-N_i(\varsigma(t))=0 \quad (16)$$

$$B_i(\varsigma(t))-F(\varsigma(t))CB_i(\varsigma(t))-G_i(\varsigma(t))=0 \quad (17)$$

$$D(\varsigma(t))-F(\varsigma(t))CD(\varsigma(t))=0 \quad (18)$$

$$v_1^T(P_i-\varepsilon_1 I)v_1 \text{ is SOS} \quad (19)$$

$$-v_2^T\Big(\sum_{k=1}^{r}\partial_k P_k+N_i^T(\varsigma(t))P_j+P_jN_i(\varsigma(t))+\varepsilon_2(\varsigma(t))I\Big)v_2 \text{ is SOS} \quad (20)$$

where v_1 and v_2 are vectors with appropriate dimensions that do not depend on $\varsigma(t)$, $\varepsilon_1>0$, and $\varepsilon_2(\varsigma(t))>0$ at $\varsigma(t)\ne 0$.

Proof:

If the conditions (16)-(18) of Theorem 1 hold, then (15) becomes

$$\dot{e}(t)=\sum_{i=1}^{r}\beta_i(\theta)\{N_i(\varsigma(t))e(t)\} \tag{21}$$

Select the non- common Lyapunov function as follows

$$V(t)=\sum_{i=1}^{r}\beta_i(\theta)e^T(t)P_ie(t) \tag{22}$$

It is noted that if the condition (19) holds, it means that the symmetric $P_i>0$.

Taking the derivative of Lyapunov function results in

$$\dot{V}(t)=\sum_{k=1}^{r}\dot{\beta}_k(\theta)e^T(t)P_ke(t)+\sum_{i=1}^{r}\beta_i(\theta)[\dot{e}^T(t)P_ie(t)+e^T(t)P_i\dot{e}(t)]. \tag{23}$$

Substituting (21) into (23), and on the basis of the Assumption 3 produces

$$\begin{aligned}\dot{V}(t)&\le\sum_{k=1}^{r}e^T(t)\partial_kP_ke(t)+\sum_{i=1}^{r}\sum_{j=1}^{r}\beta_i(\theta)\{e^T(t)N_i^T(\varsigma(t))P_je(t)\\&+e^T(t)P_jN_i(\varsigma(t))e(t)\}\le\sum_{k=1}^{r}e^T(t)\partial_kP_ke(t)+\sum_{i=1}^{r}\sum_{j=1}^{r}\beta_i(\theta)e^T(t)\\&\times\{N_i^T(\varsigma(t))P_j+P_jN_i(\varsigma(t))\}e(t)\\&\le\sum_{i=1}^{r}\sum_{j=1}^{r}\beta_i(\theta)e^T(t)\{N_i^T(\varsigma(t))P_j+P_jN_i(\varsigma(t))+\sum_{k=1}^{r}\partial_kP_k\}e(t)\end{aligned} \tag{24}$$

It can be seen from (24) that if the condition (20) holds, then $\dot{V}(t)<0$, it means that the estimation error (8) approaches zero asymptotically. The proof is completed.

Remark 5: The term positive ε_1 and $\varepsilon_2(\varsigma(t))>0$ are used in Theorem 1 to guarantee P_i is positive define matrix and $\sum_{k=1}^{r}\partial_kP_k+N^T{}_i(\varsigma(t))P_j+P_jN_i(\varsigma(t))$ is negative define matrix instead of semi-positive and semi-negative define.

In order to synthesize the observer (11), all conditions (16)-(20) must be solved to determine the parameters $N_i(\varsigma(t))$, $G_i(\varsigma(t))$, $L_i(\varsigma(t))$, and $F(\varsigma(t))$ of the observer (11). However, the condition (20) is a polynomial BMI (Bilinear Matrix Inequality) which cannot be solved by using SOS TOOL in Matlab, therefore, we need to transform it into the polynomial Linear Matrix Inequalities (LMIs). The following Theorem 2 will be with polynomial LMI.

Lemma 1 [44]: There are two matrices $A\in\Re^{m\times n}$ with $m\ge n$, $B\in\Re^{k\times n}$ and suppose that $BA^+A=B$. Then any matrix of the form $X=BA^++Y(I-AA^+)$ is a solution of $XA=B$, where $Y\in\Re^{k\times m}$ is an arbitrary matrix and A^+ is defined as $A^+=(A^TA)^{-1}A^T$ which is the Moore-Penrose pseudo-inverse of A.

Lemma 2: Let A, S, P, and R be matrices with proper sizes. The following two inequalities are equivalent:

i) $\quad R+A^TP+PA<0$

ii) $\quad \exists S:\begin{bmatrix}A^TS^T+SA+R & P-S+A^TS^T\\ P-S^T+SA & -S-S^T\end{bmatrix}<0$

Proof of Lemma 2:

***)** *ii)* implication *i)*: We pre-multiply and post-multiply (ii) with $\begin{bmatrix}I\\A\end{bmatrix}^T$ and $\begin{bmatrix}I\\A\end{bmatrix}$ respectively which yields

$$\begin{bmatrix}I\\A\end{bmatrix}^T\begin{bmatrix}A^TS^T+SA+R & P-S+A^TS^T\\ P-S^T+SA & -S-S^T\end{bmatrix}\begin{bmatrix}I\\A\end{bmatrix}=R+A^TP+PA<0$$

***)** *i)* implication *ii):* From *i)* we have

$$\begin{aligned}&(R+A^TP+PA)+(A^TS^T-A^TS^T)+(SA-SA)\\&+(A^TS^TA-A^TS^TA)+(A^TSA-A^TSA)<0\\&A^TS^T+SA+R+PA-SA+A^TS^TA+A^TP-A^TS^T\\&+A^TSA-A^TS^TA-A^TSA<0\end{aligned} \tag{*}$$

We can rewrite (*) in the form of matrix as follows

$$\begin{bmatrix}I\\A\end{bmatrix}^T\begin{bmatrix}A^TS^T+SA+R & P-S+A^TS^T\\ P-S^T+SA & -S-S^T\end{bmatrix}\begin{bmatrix}I\\A\end{bmatrix}<0$$

That leads to

$$\begin{bmatrix}A^TS^T+SA+R & P-S+A^TS^T\\ P-S^T+SA & -S-S^T\end{bmatrix}<0$$

Therefore *i*) and *ii*) are equivalent. The proof is completed.

Theorem 2: Under Assumptions 1, 2, and 3, the estimation error (8) with the observer (11) converges to zero asymptotically if there exist polynomial matrices $F(\varsigma(t))$, $N_i(\varsigma(t))$, $L_i(\varsigma(t))$, $G_i(\varsigma(t))$, $K(\varsigma(t))$, $X(\varsigma(t))$, $Q_i(\varsigma(t))$, and symmetric matrix P_i such that the following conditions hold for *i*=1, 2, …, *r*.

$$v_1^T(P_i-\varepsilon_1I)v_1 \text{ is SOS} \tag{25}$$

$$-v_2^T\left(\begin{bmatrix}\Phi_{ik}^{(11)} & (*)\\ \Omega_{ij}^{(12)} & -X(\varsigma(t))-X^T(\varsigma(t))\end{bmatrix}+\varepsilon_2(\varsigma(t))I\right)v_2 \text{ is SOS} \tag{26}$$

$$\Phi_{ik}^{(11)} = \sum_{k=1}^{r} \partial_k P_k + (A_i(\varsigma(t)) - U(\varsigma(t))CA_i(\varsigma(t)))^T X^T(\varsigma(t))$$
$$-(V(\varsigma(t))CA_i(\varsigma(t)))^T K^T(\varsigma(t)) - C^T Q_i^T(\varsigma(t)) + X(\varsigma(t))(A_i(\varsigma(t))$$
$$-U(\varsigma(t))CA_i(\varsigma(t))) - K(\varsigma(t))(V(\varsigma(t))CA_i(\varsigma\xi(t))) - Q_i(\varsigma(t))C \quad (27)$$

$$\Omega_{ij}^{(12)} = P_j - X^T(\varsigma(t)) + X(\varsigma(t))(A_i(\varsigma(t)) - U(\varsigma(t))CA_i(\varsigma(t)))$$
$$-K(\varsigma(t))(V(\varsigma(t))CA_i(\varsigma(t))) - Q_i(\varsigma(t))C \quad (28)$$

where v_1 and v_2 are vectors with appropriate dimensions that do not depend on $\varsigma(t)$, $\varepsilon_1 > 0$, $\varepsilon_2(\varsigma(t)) > 0$ at $\varsigma(t) \neq 0$, and

$$(CD(\varsigma(t)))^+ = ((CD(\varsigma(t)))^T (CD(\varsigma(t))))^{-1} (CD(\varsigma(t)))^T$$

$$U(\varsigma(t)) = D(\varsigma(t))(CD(\varsigma(t)))^+ \quad (29)$$

$$V(\varsigma(t)) = (I - (CD(\varsigma(t)))(CD(\varsigma(t)))^+) \quad (30)$$

$$K(\varsigma(t)) = X(\varsigma(t))Y(\varsigma(t)) \quad (31)$$

$$Q_i(\varsigma(t)) = X(\varsigma(t))L_i(\varsigma(t)). \quad (32)$$

Moreover, the parameters of observer (11) are computed as follows

$$F(\varsigma(t)) = U(\varsigma(t)) + Y(\varsigma(t))V(\varsigma(t)) \quad (33)$$

$$G_i(\varsigma(t)) = B_i(\varsigma(t)) - F(\varsigma(t))CB_i(\varsigma(t)) \quad (34)$$

$$L_i(\varsigma(t)) = X^{-1}(\varsigma(t))\mathrm{Q}_i(\varsigma(t)) \quad (35)$$

$$N_i(\varsigma(t)) = A_i(\varsigma(t)) - F(\varsigma(t))CA_i(\varsigma(t)) - L_i(\varsigma(t))C \quad (36)$$

Proof:
From the condition (20) of Theorem 1, it can be inferred that

$$\sum_{k=1}^{r} \partial_k P_k + N_i^T(\varsigma(t))P_j + P_j N_i(\varsigma(t)) < 0. \quad (37)$$

Employing the Lemma 2 for (37) with slack variable $X(\varsigma(t))$, one obtains

$$\begin{bmatrix} \Phi_{ik}^{(11)} & (*) \\ \Omega_{ij}^{(12)} & -X(\varsigma(t)) - X^T(\varsigma(t)) \end{bmatrix} < 0 \quad (38)$$

where

$$\Phi_{ik}^{(11)} = \sum_{k=1}^{r} \partial_k P_k + N_i^T(\varsigma(t))X^T(\varsigma(t)) + X(\varsigma(t))N_i(\varsigma(t)) \quad (39)$$

$$\Omega_{ij}^{(12)} = P_j - X^T(\varsigma(t)) + X(\varsigma(t))N_i(\varsigma(t)). \quad (40)$$

From (18), we have

$$F(\varsigma(t))CD(\varsigma(t)) = D(\varsigma(t)) \quad (41)$$

On the basis of the Lemma 1, the general solution of (41) is

$$F(\varsigma(t)) = D(\varsigma(t))(CD(\varsigma(t)))^+ + Y(\varsigma(t))(I - (CD(\varsigma(t)))(CD(\varsigma(t)))^+) \quad (42)$$

where

$$(CD(\varsigma(t)))^+ = ((CD(\varsigma(t)))^T (CD(\varsigma(t))))^{-1} (CD(\varsigma(t)))^T.$$

It is noted that the existence of the general solution (42) is guaranteed if only if the matrices C and $D(\varsigma(\mathrm{t}))$ satisfy the Assumption 2.
Let's denote

$$U(\varsigma(t)) = D(\varsigma(t))(CD(\varsigma(t)))^+ \quad (43)$$

$$V(\varsigma(t)) = (I - (CD(\varsigma(t)))(CD(\varsigma(t)))^+). \quad (44)$$

Substituting (43) and (44) into (42) yields

$$F(\varsigma(t)) = U(\varsigma(t)) + Y(\varsigma(t))V(\varsigma(t)). \quad (45)$$

From (16) and (45), one obtains

$$N_i(\varsigma(t)) = A_i(\varsigma(t)) - (U(\varsigma(t)) + Y(\varsigma(t))V(\varsigma(t)))CA_i(\varsigma(t))$$
$$-L_i(\varsigma(t))C = A_i(\varsigma(t)) - (U(\varsigma(t))CA_i(\varsigma(t))) \quad (46)$$
$$-(Y(\varsigma(t))V(\varsigma(t))CA_i(\varsigma(t))) - L_i(\varsigma(t))C$$

Substituting (46) into (39), the obtained result is

$$\Phi_{ik}^{(11)} = \sum_{k=1}^{r} \partial_k P_k + (A_i(\varsigma(t)) - (U(\varsigma(t))CA_i(\varsigma(t))) - (Y(\varsigma(t))$$
$$\times V(\varsigma(t))CA_i(\varsigma(t))) - L_i(\varsigma(t))C)^T X^T(\varsigma(t)) + X(\varsigma(t))(A_i(\varsigma(t))$$
$$-(U(\varsigma(t))CA_i(\varsigma(t))) - (Y(\varsigma(t))V(\varsigma(t))CA_i(\varsigma(t))) - L_i(\varsigma(t))C)$$
$$= \sum_{k=1}^{r} \partial_k P_k + (A_i(\varsigma(t)) - U(\varsigma(t))CA_i(\varsigma(t)))^T X^T(\varsigma(t))$$
$$-(V(\varsigma(t))CA_i(\varsigma(t)))^T Y^T(\varsigma(t))X^T(\varsigma(t)) - C^T L_i^T(\varsigma(t))X^T(\varsigma(t))$$
$$+X(\varsigma(t))(A_i(\varsigma(t)) - U(\varsigma(t))CA_i(\varsigma(t)))$$
$$-X(\varsigma(t))Y(\varsigma(t))(V(\varsigma(t))CA_i(\varsigma(t))) - X(\varsigma(t))L_i(\varsigma(t))C. \quad (47)$$

Substituting (46) into (40), we have

$$\Omega_{ij}^{(12)} = P_j - X^T(\varsigma(t)) + X(\varsigma(t))(A_i(\varsigma(t)) - (U(\varsigma(t))CA_i(\varsigma(t)))$$
$$-(Y(\varsigma(t))V(\varsigma(t))CA_i(\varsigma(t))) - L_i(\varsigma(t))C)$$
$$= P_j - X^T(\varsigma(t)) + X(\varsigma(t))(A_i(\varsigma(t)) - U(\varsigma(t))CA_i(\varsigma\xi(t))) \quad (48)$$
$$-X(\varsigma(t))Y(\varsigma(t))(V(\varsigma(t))CA_i(\varsigma(t))) - X(\varsigma(t))L_i(\varsigma(t))C$$

Let us define

$$K(\varsigma(t)) = X(\varsigma(t))Y(\varsigma(t)) \quad (49)$$

$$\mathrm{Q}_i(\varsigma(t)) = X(\varsigma(t))L_i(\varsigma(t)) \quad (50)$$

Substituting (49) and (50) into (47) and (48), we obtain

$$\Phi_{ik}^{(11)} = \sum_{k=1}^{r}\partial_k P_k + (A_i(\varsigma(t)) - U(\varsigma(t))CA_i(\varsigma(t)))^T X^T(\varsigma(t))$$
$$-(V(\varsigma(t))CA_i(\varsigma(t)))^T K^T(\varsigma(t) - C^T Q_i^T(\varsigma(t)) + X(\varsigma(t))(A_i(\varsigma(t))$$
$$-U(\varsigma(t))CA_i(\varsigma(t))) - K(\varsigma(t))(V(\varsigma(t))CA_i(\varsigma(t))) - Q_i(\varsigma(t))C \quad (51)$$

$$\Omega_{ij}^{(12)} = P_j - X^T(\varsigma(t)) + X(\varsigma(t))(A_i(\varsigma(t)) - U(\varsigma(t))CA_i(\varsigma(t)))$$
$$-K(\varsigma(t))(V(\varsigma(t))CA_i(\varsigma(t))) - Q_i(\varsigma(t))C. \quad (52)$$

From (51) and (52), it is seen that (38) becomes

$$\begin{bmatrix} \Phi_{ik}^{(11)} & (*) \\ \Omega_{ij}^{(12)} & -X(\xi(t)) - X^T(\xi(t)) \end{bmatrix} < 0 \quad (53)$$

where $\Phi_{ik}^{(11)}$ and $\Omega_{ij}^{(12)}$ are expressed as (51) and (52), respectively.

It is seen that (53) is equivalent to (26) of Theorem 2 and it is a polynomial LMI. It means that the polynomial BMI (20) has been successfully transformed into polynomial LMI in Theorem 2. The proof is completed.

It is noted that the polynomial LMI in Theorem 2 can be solved easily by using SOS TOOL of Matlab [46]. The brief procedure for the observer (11) synthesis is presented below:

Step 1: Check the matrices C and $D(\varsigma(t))$ satisfy the Assumption 2 or not**.** If yes, we go to the next step. If not, this method does not work for this case.

Step 2: From (29) and (30), $U(\varsigma(t))$ and $V(\varsigma(t))$ are obtained.

Step 3: Resolve (25) and (26) to obtain P_j, $K(\varsigma(t))$, $X(\varsigma(t))$, $Q_i(\varsigma(t))$, then from (31) we obtain $Y(\varsigma(t))$.

Step 4: The matrices $F(\varsigma(t))$, $G_i(\xi(t))$, $L_i(\varsigma(t))$ and $N_i(\varsigma(t))$ are obtained from (33)-(36), respectively.

Step 5: The observer (11) is synthesized.

IV. Illustrative Examples

In this section, two examples are illustrated to prove the effectiveness of the proposed method. Example 1 is a numerical example and Example 2 is an application for Inverted Pendulum.

A. Example 1

Consider a nonlinear system as follows

$$\begin{cases} \dot{x}_1(t) = \cos(x_1(t))x_2(t) + x_1^{\,3}(t)u(t) \\ \dot{x}_2(t) = x_1(t) - x_1^{\,2}(t)x_2(t) \\ y(t) = x_1(t) \end{cases} \quad (54)$$

Let the system (54) be transformed into a polynomial T-S fuzzy system where $\varsigma(t)$ of polynomial T-S fuzzy system is the output *y(t).* Then

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2}\beta_i(\theta(t))[A_i(y)x(t) + B_i(y)u(t)] \\ y(t) = Cx(t) \end{cases} \quad (55)$$

$$A_1(y) = \begin{bmatrix} 0 & 1 \\ 1 & -y^2 \end{bmatrix},\ A_2(y) = \begin{bmatrix} 0 & -1 \\ 1 & -y^2 \end{bmatrix},\ B_1(y) = B_2(y) = \begin{bmatrix} y^3 \\ 0 \end{bmatrix}$$

, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

$$\beta_1(\theta(t)) = \frac{1+\cos(x_1(t))}{2}; \beta_2(\theta(t)) = \frac{1-\cos(x_1(t))}{2}.$$

We can see that

$\left|\dot{\beta}_1(\theta(t))\right| = \left|-\dot{x}_1\dfrac{\sin(x_1)}{2}\right|, \left|\dot{\beta}_2(\theta(t))\right| = \left|\dot{x}_1\dfrac{\sin(x_1)}{2}\right|$, then we assume $\partial_1 = \partial_2 = 0.4$.

Suppose the system (55) is influenced by the uncertainties and the bounded constraints of these uncertainties are unknown. The system (55) is expressed in the following form

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2}\beta_i(\theta(t))[(A_i(y) + \Delta A_i(y))x(t) + (B_i(y) + \Delta B_i(y))u(t)] \\ y(t) = Cx(t) \end{cases} \quad (56)$$

where

$$\Delta A_1(y) = \begin{bmatrix} \cos(2y)/y & -2\sin(2y+1)/y \\ 0.25\cos(2y) & -0.5\sin(2y+1) \end{bmatrix},$$
$$\Delta A_2(y) = \begin{bmatrix} -\sin(2y)/y & -2.4\cos(y)/y \\ -0.25\sin(2y) & -0.6\cos(y) \end{bmatrix},$$
$$\Delta B_1(y) = \begin{bmatrix} 2.2\cos(y^2)/y \\ 0.55\cos(y^2) \end{bmatrix}, \Delta B_2(y) = \begin{bmatrix} \sin(y^2)/y \\ 0.25\sin(y^2) \end{bmatrix}.$$

Transforming system (56) into the unknown input system yields

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2}\beta_i(\theta(t))[A_i(y)x(t) + B_i(y)u(t) + D(y)\omega_i(y)] \\ y(t) = Cx(t) \end{cases} \quad (57)$$

where

$$D(y) = \begin{bmatrix} 2/y \\ 0.5 \end{bmatrix}.$$

Step 1: We can see that C, $D(y(t))$ are full row and column ranks**,** and $\text{normal rank}(CD(\varsigma(t))) = \text{normal rank}(D(\varsigma(t)) = 1$, thus the Assumption 2 is satisfied.

Step 2: From (29) and (30),

$$U(y)=\begin{bmatrix}1\\ y/4\end{bmatrix},\ V(y)=0\,;$$

Step 3: Resolve the constraint (25) and (26) of Theorem 2 using the SOS TOOL of Matlab. We can obtain the values below

$$P_1=\begin{bmatrix}1.0397 & -0.24409\times10^{-5}\\ -0.24409\times10^{-5} & 0.13246\times10^{-8}\end{bmatrix},$$

$$P_2=\begin{bmatrix}1.0369 & -0.24294\times10^{-5}\\ -0.24294\times10^{-5} & 0.11553\times10^{-8}\end{bmatrix},$$

$$K(y)=\begin{bmatrix}0\\ 0\end{bmatrix},\ Y(y)=\begin{bmatrix}0\\ 0\end{bmatrix},$$

$$X=\begin{bmatrix}0.0013842 & 0.20861\times10^{-10}\\ -0.32898\times10^{-5} & 0.33362\times10^{-8}\end{bmatrix},$$

$$Q_1(y)=\begin{bmatrix}0.2772\times10^{-8}y^2-0.6543\times10^{-8}y+1.0351\\ -0.5871\times10^{-10}y^2-0.2509\times10^{-7}y-0.2135\times10^{-5}\end{bmatrix},$$

$$Q_2(y)=\begin{bmatrix}0.27083\times10^{-8}y^2+0.78949\times10^{-9}y+1.0377\\ 0.16966\times10^{-9}y^2+0.39287\times10^{-6}y+0.34\times10^{-5}\end{bmatrix}.$$

Step 4 and Step 5: The matrix $F(\varsigma(t))$ is obtained from (33). We obtain $N_i(\varsigma(t))$ from (36), $L_i(\varsigma(t))$ from (35) and $G_i(\xi(t))$ from (34).

$$N_1(y)=\begin{bmatrix}-0.2003\times10^{-5}y^2+0.4613\times10^{-5}y & 0\\ -747.7854 & \\ 0.0156y^2+7.52y-736744.09 & -y^2-y/4\end{bmatrix}$$

$$N_2(y)=\begin{bmatrix}-0.1955\times10^{-5}y^2+0.120436\times10^{-5}y & 0\\ -749.6637 & \\ -0.0527y^2-117.7585y-740255.5170 & -y^2+y/4\end{bmatrix}$$

$$L_1(y)=\begin{bmatrix}0.200349\times10^{-5}y^2-0.461377\times10^{-5}y+747.785457\\ -0.015625y^2-7.526880y+1736745.098989\end{bmatrix},$$

$$L_2(y)=\begin{bmatrix}0.195579\times10^{-5}y^2-0.120435\times10^{-5}y+749.663746\\ 0.052783y^2+117.758539y+740256.517083\end{bmatrix}$$

$$G_1(y)=\begin{bmatrix}0\\ -y^4/4\end{bmatrix},\ G_2(y)=\begin{bmatrix}0\\ -y^4/4\end{bmatrix},\ F(y)=\begin{bmatrix}1\\ y/4\end{bmatrix}.$$

Here, we used Simulink tools to carry out the simulation. The initial states of the system $x(0)=[-1.5\ \ -30]^T$ and the estimated states are $\hat{x}(0)=[-0.5\ \ -50]^T$. The input used for simulation is $u(t)=0.2\sin(t)$.

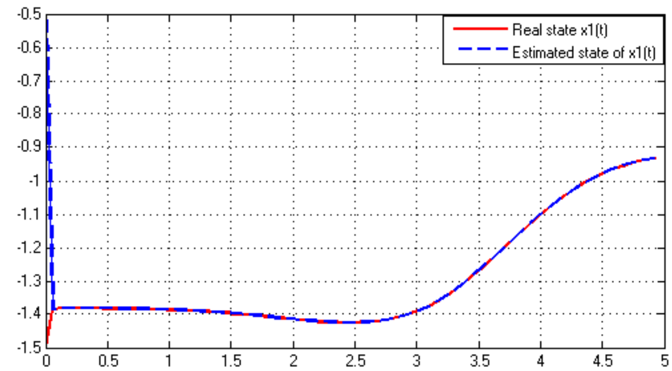

Fig. 1. Real state $x_1(t)$ and estimated state $\hat{x}_1(t)$.

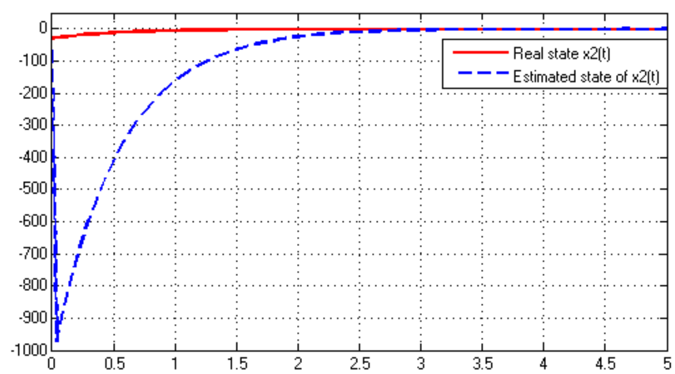

Fig. 2. Real state $x_2(t)$ and estimated state $\hat{x}_2(t)$.

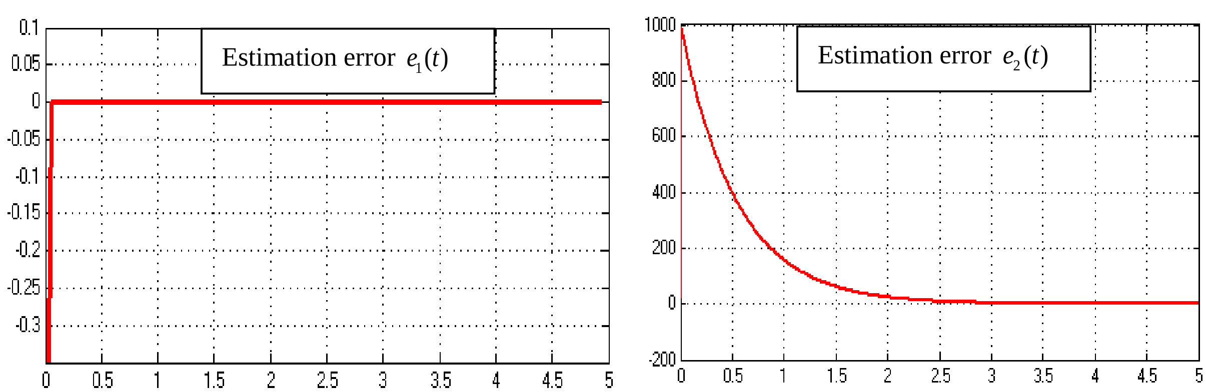

Fig. 3. Estimation errors $e_1(t)$ and $e_2(t)$

Figures 1-3 show the simulation results of the numerical example, in which the system (56) is perturbed by the uncertainties $\Delta A_1(y(t))$, $\Delta A_2(y(t))$, $\Delta B_1(y(t))$, and $\Delta B_2(y(t))$. Figures 1 and 2 depict the states x_1, x_2 and their estimated signals. The estimation error is illustrated in Fig. 3. It can be seen from these figures that the estimated states can approach real states asymptotically. Hence, the proposed method is successful in synthesizing observer for polynomial T-S fuzzy system with uncertainties.

Remark 6: It is noted that if we use general T-S fuzzy system to represent the nonlinear system (54), both $\cos(x_1)$ and x_1 must be linearized and the local bound range of x_1 must be given. Then there will be four fuzzy rules in the T-S fuzzy system. However, if let a polynomial T-S fuzzy system be present the original nonlinear system, x_1 will be putted in the system matrices, therefore the bound range of x_1 does not need to know and the number of fuzzy rules will be reduced to two. It means that the polynomial T-S fuzzy system will represent the system (54) be more exactly over global region.

B. Example 2

In this example, we consider a practical dynamic model of an Inverted Pendulum on a cart [21]. The system is depicted in the following figure.

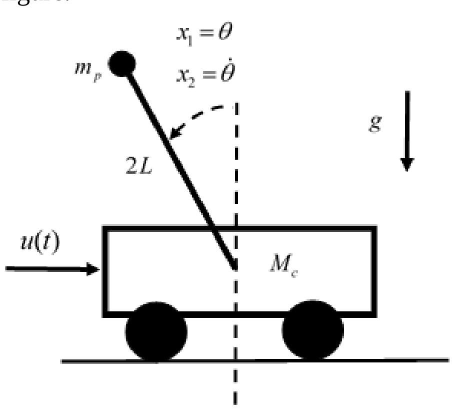

Fig. 4. Inverted Pendulum on a cart

The nonlinear equation of the Inverted Pendulum on a cart is expressed as follows

$$\dot{x}_1(t)=x_2(t)$$
$$\dot{x}_2(t)=\frac{g\sin(x_1(t))-am_pLx_2^2(t)\sin(x_1(t))\cos(x_1(t))}{4L/3-am_pL\cos^2(x_1(t))}-\frac{a\cos(x_1(t))u}{4L/3-am_pL\cos^2(x_1(t))} \quad (58)$$

where $x=\begin{bmatrix}x_1 & x_2\end{bmatrix}=\begin{bmatrix}\theta & \dot{\theta}\end{bmatrix}$ is a vector of states, θ is the angle (in radians), $\dot{\theta}$ is the angular velocity. $m_p=2kg$, $M_c=8kg$, $2L=1m$ are the mass of Pendulum, Cart and the length of the pendulum, respectively; $a=1/(m_p+M_c)$. $u(t)$ is the control input force imposed on the cart, and $x_1(t)\in\left[-\frac{70\pi}{180} \quad \frac{70\pi}{180}\right]$. In order to reduce computational burden, the term $\sin(x_1(t))\approx s_1x_1(t)$, $\tan(x_1(t))\approx t_1x_1(t)$, $s_1=0.8578$ and $t_1=1.5534$.

Applying nonlinear sector to linearize this system, the nonlinear system (58) is represented as the following polynomial T-S fuzzy system. It is noted that $y=Cx(t)=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_2$, therefore, the result of the nonlinear system can be expressed in the form of (3) with $\varsigma(t)$ being the output *y(t).*

$$\begin{cases}\dot{x}(t)=\sum_{i=1}^{2}\beta_i(x_1(t))\left[A_i(y)x(t)+B_i(y)u(t)\right]\\ y=Cx(t)\end{cases} \quad (59)$$

where

$$A_1(y)=\begin{bmatrix}0 & 1\\ a_1(y) & 0\end{bmatrix},\ A_2(y)=\begin{bmatrix}0 & 1\\ a_2(y) & 0\end{bmatrix},\ B_1=\begin{bmatrix}0\\ -f_{1\min}a\end{bmatrix},$$
$$B_2=\begin{bmatrix}0\\ -f_{1\max}a\end{bmatrix},\ C=\begin{bmatrix}0 & 1\end{bmatrix},$$
$$f_{1\min}=0.5222,\ f_{1\max}=1.7647$$
$$a_1(y)=f_{1\min}(gt_1-am_pLy^2s_1),\ a_2(y)=f_{1\max}(gt_1-am_pLy^2s_1)$$

The premise variables

$$\beta_1(x_1(t))=\frac{f_1(x_1)-f_{1\max}}{f_{1\min}-f_{1\max}},\ \beta_2(x_1(t))=1-\beta_1(x_1(t));$$
$$f_1(x_1)=\frac{\cos(x_1)}{4L/3-am_pL\cos^2(x_1)}$$

Suppose the (59) are affect by the following uncertainties, then

$$\begin{cases}\dot{x}(t)=\sum_{i=1}^{2}\beta_i(x_1(t))\left[(A_i(y)+\Delta A_i(y))x(t)+(B_i(y)+\Delta B_i(y))u(t)\right]\\ y=Cx(t)\end{cases} \quad (60)$$

$$\Delta A_1(y)=\begin{bmatrix}0.21y^2\sin(2y+1) & -0.12y^2\cos(y)\\ 0.55\sin(2y+1) & -0.6\cos(y)\end{bmatrix},$$
$$\Delta A_2(y)=\begin{bmatrix}-0.11y^2\cos(2y) & 0.02y^2\sin(y)\\ -0.55\cos(2y) & 0.1\sin(y)\end{bmatrix},$$

$$\Delta B_1(y)=\begin{bmatrix}0.21y^2\cos(y^2)\\1.05\cos(y^2)\end{bmatrix},\ \Delta B_2(y)=\begin{bmatrix}0.21y^2\sin(y^2)\\1.05\sin(y^2)\end{bmatrix}.$$

Transform (60) into unknown input system with $D(y)=\begin{bmatrix}0.1y^2\\0.5\end{bmatrix}$. With $x_1(t)\in\left[-\frac{70\pi}{180}\ \ \frac{70\pi}{180}\right]$, in this paper, we assume $\partial_1=\partial_2=2.5$.

Step 1: It is seen that the matrices C, $D(y(t))$ are full row and column ranks, respectively, and $\text{normal rank}(CD(\varsigma(t)))=\text{normal rank}(D(\varsigma(t))=1$, therefore, the Assumption 2 is satisfied.

Step 2: The matrices $U_i(y)$ and $V_i(y)$ are calculated:

$$U(y)=\begin{bmatrix}y^2/5\\1\end{bmatrix},\ V(y)=0\,.$$

Step 3: The matrices P_i, $Q_i(y(t))$, $K(y(t))$, $Y(y(t))$, and $X(y(t))$ are obtained as follows

$$P_1=\begin{bmatrix}0.40015\times10^{-8} & 0.19773\times10^{-10}\\0.19773\times10^{-10} & 0.20074\times10^{-8}\end{bmatrix},$$

$$P_2=\begin{bmatrix}0.40092\times10^{-8} & 0.70957\times10^{-10}\\0.70957\times10^{-10} & 0.34511\times10^{-8}\end{bmatrix},Y(y)=\begin{bmatrix}0\\0\end{bmatrix},$$

$$K(y)=\begin{bmatrix}0\\0\end{bmatrix},\ X(y)=\begin{bmatrix}0.28496\times10^{-7} & 0.15309\times10^{-4}\\0.2297\times10^{-8} & 0.43039\end{bmatrix},$$

$$Q_1(y)=\begin{bmatrix}-0.8162\times10^{-10}y^2-0.232\times10^{-11}y+0.153\times10^{-4}\\0.69247\times10^{-4}y^2-0.40629\times10^{-10}y+0.43036\end{bmatrix}$$

$$Q_2(y)=\begin{bmatrix}0.883\times10^{-9}y^2+0.985\times10^{-16}y+0.153\times10^{-4}\\0.702\times10^{-4}y^2-0.881\times10^{-10}y+0.430\end{bmatrix}.$$

Step 4 and Step 5: The parameters $N_i(y)$, $G_i(y)$, $F(y)$, and $L_i(y(t))$ are computed

The matrices $N_1(y)$ and $N_2(y)$ are expressed in (61) and (62), respectively,

$$F(y)=\begin{bmatrix}y^2/5\\1\end{bmatrix},\ G_1(y)=\begin{bmatrix}(261y^2)/25000\\0\end{bmatrix},$$

$$G_2(y)=\begin{bmatrix}(7767y^2)/250000\\0\end{bmatrix},$$

$$L_1(y)=\begin{bmatrix}-0.089302y^2-0.8\times10^{-4}y+1.055137\\0.161\times10^{-3}y^2-0.939658\times10^{-10}y+0.999930\end{bmatrix},$$

$$L_2(y)=\begin{bmatrix}-0.056656y^2+0.455986\times10^{-8}y+1.032527\\0.163\times10^{-3}y^2-0.204865\times10^{-11}y+0.999907\end{bmatrix}.$$

After obtaining all parameters, the observer (11) is constructed. The simulation results of the fuzzy observer for the system in Example 2 with input $u(t)=0.1\sin(t)$ and initial values of the states $x(0)=[70*\text{pi}/180\quad 1]^T$, estimated states $\hat{x}(0)=[\text{-}35*\text{pi}/180\quad -1.2]^T$ are illustrated as follows.

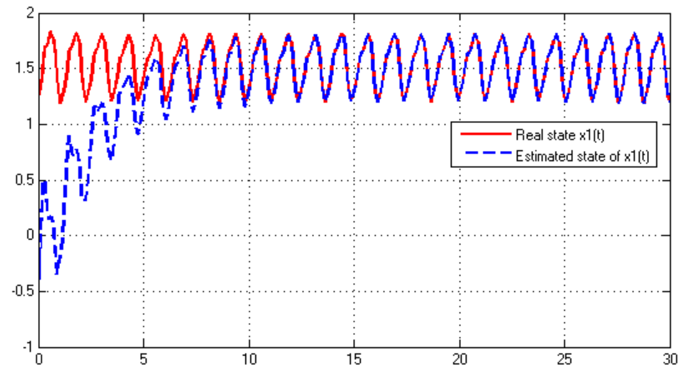

Fig. 5. Real state $x_1(t)$ and estimated state $\hat{x}_1(t)$

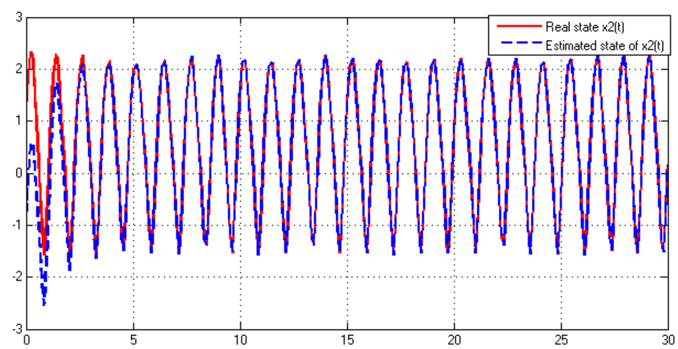

Fig. 6. Real state $x_2(t)$ and estimated state $\hat{x}_2(t)$

The simulation results of the observer synthesis for the practical dynamic system of Inverted Pendulum on a Cart which is affected by uncertainties are illustrated in Figs. 5-7. Obviously, Fig. 5 and Fig. 6 show that estimation states $\hat{x}_1$ and

$$N_1(y)=\begin{bmatrix}(y^2((0.0447y^2)-7.9496))/5 & 0.0893y^2+0.81\times10^{-4}y-0.0551\\0 & -0.161\times10^{-3}y^2+0.9397\times10^{-10}y-0.999\end{bmatrix}\quad(61)$$

$$N_2(y)=\begin{bmatrix}(y^2((0.1513y^2)-26.8645))/5 & 0.0566y^2-0.4559\times10^{-8}y-0.0325\\0 & -0.163\times10^{-3}y^2+0.2048\times10^{-11}y-0.9999\end{bmatrix}\quad(62)$$

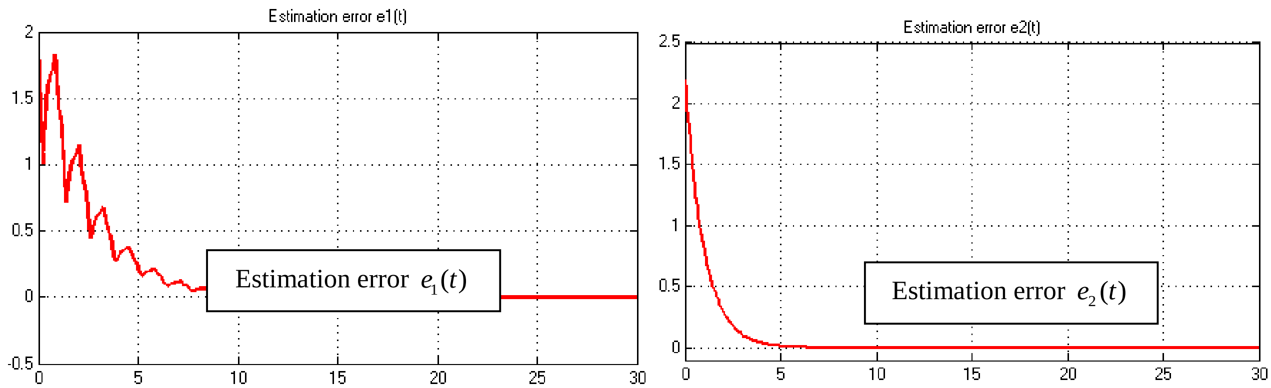

Fig. 7. Estimation errors $e_1(t)$ and $e_2(t)$

$\hat{x}_2$, approach asymptotically to real states x_1 and x_2, respectively. In Fig. 7, it is clearly seen that the estimation errors e_1 and e_2 converge to zero asymptotically.

Remark 7: $\left|\dot{\beta}_k(\theta(t))\right| \le \partial_k$, $k = 1, 2, ..., r$, in Assumption 3 is used in the above two examples. However, in this study, the bound range of some or all state variables are not known, and the bound of the uncertainties is not known either; hence, the upper bounds ∂_1 and ∂_2 are difficult to estimate. Therefore, in this study, the values of ∂_1 and ∂_2 in Example 1 and 2 are selected by trial and error. If the chosen ∂_1 and ∂_2 make the observer synthesis be feasible and the simulation is successful, then the synthesized observer is effective. If the chosen ∂_1 and ∂_2 either make the observer synthesis be not feasible or the simulation is not successful, we need to choose another couple ∂_1 and ∂_2. Based on our experience, if the feasible observer is not obtained, we may decrease ∂_1 and/or ∂_2 to design it again; however, if the observer synthesis is feasible, but the simulation is not successful, we may increase ∂_1 and/or ∂_2. We have to admit that trial and error method is not reliable for the observer synthesis, it should be resolved in the future work.

Remark 8: It is worth noting that we assume that the upper bounds of the uncertainties in Example 1 and Example 2 are unavailable, therefore, the method in [21] and [28] will fail to design observer for the system (56) and (60). By using the proposed method with a new form of the observer (12), the observer has been synthesized successfully for the uncertain polynomial T-S fuzzy system, in which the bounded constraints are unknown and the controller did not need to be designed together with an observer for the purpose of eliminating the effects of uncertainties.

V. Conclusion

This paper has proposed a new approach based on the unknown input method to synthesize the observer for the uncertain polynomial T-S fuzzy system. On the basis of non-common Lyanpunov function and SOS technique, two main theorems which contain the conditions for observer synthesis have been derived. These conditions are solved easily by using SOS TOOL of Matlab. Finally, two examples have demonstrated to present the effectiveness of the proposed method. The main characteristics of this paper are as follows. (i) There are no upper bound limits for uncertainties. (ii) The observer synthesis can be completed independently without designing controller together. (iii) The observer form is new and is notably completely different from other existing traditional form of observers.

Acknowledgment

This work was supported by the Grant MOST104-2221-E-008-054-MY3 from the Ministry of Science and Technology of Taiwan and the corresponding author Dr. W. J. Wang completed this work when he was the visiting scholar in the University of Louisville, KY with the Grant MOST 105-2918-I-008-007.

References

[1] T. Takagi and M. Sugeno, “Fuzzy identification of systems and its application to modeling and control,” *IEEE Trans. Syst. Man Cybern.*, vol. 15, no. 1, pp. 116-132, 1985.
[2] K. Tanaka and M. Sugeno, “Stability analysis and design of fuzzy control systems,” *Fuzzy Sets Syst.*, vol. 45, no. 2, pp. 135-156, Jan. 1992.
[3] H. G. Zhang and Y. B. Quan, “Modeling, identification, and control of a class of nonlinear systems,” *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 2, pp. 349-354, 2001.
[4] H. G. Zhang and D. R. Liu, *Fuzzy modelling and fuzzy control,* Birkhauser Boston, 2006.
[5] H. G. Zhang, J. L. Zhang, G. H. Yang, and Y. H. Luo, “Leader-based optimal coordination control for the consensus problem of multiagent differential games via fuzzy adaptive dynamic programming,” *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 1, pp. 152-163, 2015.
[6] O. Castillo and P. Melin, *Type-2 fuzzy logic: theory and applications*, vol. 223, Springer-Verlag Berlin Heidelberg, 2008.
[7] H. Y. Li, J. H. Wang, H. K. Lam, Q. Zhou, and H. P. Du, “Adaptive sliding mode control for interval type-2 fuzzy systems,” *IEEE Trans. Syst., Man, Cyber.:Syst.*., vol. 46, no. 12, pp. 1654-1663, 2016.
[8] K. Tanaka, H. Yoshida, H. Ohtake, and Hua O. Wang, “A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems,” *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 4, pp. 911-922, 2009.
[9] B. Xiao, H.K. Lam, and H. Y. Li, “Stabilization of interval type-2 polynomial-fuzzy-model-based control systems,” *IEEE Trans. Fuzzy Syst.*, vol. 25, no. 1, pp. 205-217, 2017.
[10] A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, and P. A. Parrilo, *SOSTOOLS: Sum of Squares Optimization Toolbox for MATLAB*, Version 3.01, 2016.

[11] K. Tanaka, H. Ohtake, and H. O. Wang, "Guaranteed cost control of polynomial fuzzy systems via a sum of squares approach," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 39, no. 2, pp. 561-567, 2009.
[12] M. Narimani and H. K. Lam, "SOS-based stability analysis of polynomial fuzzy-model-based control systems via polynomial membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 5, pp. 862-871, Oct. 2010.
[13] H. K. Lam, M. Narimani, H. Y. Li, and H. H. Liu, "Stability analysis of polynomial-fuzzy-model-based control systems using switching polynomial Lyapunov function," *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 5, pp. 800-813, 2013.
[14] K. Guelton, N. Manamanni, C. C. Duong, and D. L. Koumba-Emianiwe, "Sum-of-squares stability analysis of Takagi-Sugeno systems based on multiple polynomial Lyapunov functions," *International Journal of Fuzzy Systems*, vol. 15, no. 1, pp. 1-8, March 2013.
[15] G. R. Yu, L. W. Huang, and C. Y. Cheng, "The SOS-based extended design of polynomial fuzzy control," *the IEEE International Conference on Systems, Man, and Cybernetics*, October, 2012, Seoul, Korea, pp. 2463-2468.
[16] H. K. Lam and S. H. Tsai, "Stability analysis of polynomial-fuzzy-model-based control systems with mismatched premise membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 1, pp. 223-229, 2014.
[17] J. L. Pitarch, A. Sala, and C. V. Arino, "Closed-form estimates of the domain of attraction for nonlinear systems via fuzzy-polynomial models," *IEEE Trans. Fuzzy Syst*., vol. 44, no. 4, pp. 526-538, 2014.
[18] R. Furqon, Y. J. Chen, M. Tanaka, K. Tanaka, and H. O. Wang, " An SOS-based control Lyapunov function design for polynomial fuzzy control of nonlinear systems," *IEEE Trans. Fuzzy Syst*., vol. 13, no. 9, pp. 1-13, 2014.
[19] Y. J. Chen, M. Tanaka, K. Tanaka, H. Ohtake, and H. O. Wang, "Discrete polynomial fuzzy systems control," *IET Control Theory Appl.,* vol. 8, no. 4, pp. 288–296, 2014.
[20] Y. Y. Wang, H. G. Zhang, Y. C. Wang, and J. Y. Zhang, "Stability analysis and controller design of discrete-time polynomial fuzzy time-varying delay systems," *Journal of the Franklin Institute,* vol. 352, pp. 5661–5685, 2015.
[21] C. Liu and H. K. Lam, " Design of polynomial fuzzy observer-controller with sampled-output measurements for nonlinear systems considering unmeasurable premise variables," *IEEE Trans. Fuzzy Syst.,*vol. 23, no. 6, pp. 2067-2079, 2015.
[22] H. K Lam, L. G. Wu, and J. Lam, "Two-step stability analysis for general polynomial-fuzzy-model-based control systems," *IEEE Trans. Fuzzy Syst*., vol. 23, no. 3, pp. 511-524, 2015.
[23] C. Liu, H. K. Lam, X. J. Ban, and X. D. Zhao, "Design of polynomial fuzzy observer-controller with membership function using unmeasurable premise variables for nonlinear systems," *Information Sciences*, vol. 355-356, pp. 186-207, 2016.
[24] H. G. Han , J. Y. Chen , and H. R. Karimi, "State and disturbance observers-based polynomial fuzzy controller," *Information Sciences*, vol. 382–383, pp. 38–59, 2017.
[25] K. Tanaka, H. Ohtake, T. Seo, and H. O. Wang, "An SOS-based observer design for polynomial fuzzy systems," *the 2011 American Control Conference on O'Farrell Street*, San Francisco, CA, USA, June 29 - July 01, 2011, pp. 4953-4958.
[26] K. Tanaka, H. Ohtake, M. Wada, H. O. Wang, and Y. J. Chen, "Polynomial fuzzy observer design: a sum of squares approach," *the Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, P.R. China, Dec. 2009, pp. 7771-7776.
[27] K. Tanaka, H. Ohtake, T. Seo, M. Tanaka, and H.O. Wang, "Polynomial fuzzy observer designs: a sum-of-squares approach," *IEEE Trans. Syst., Man, and Cybern.—Part B: Cybern*., vol. 42, no. 5, pp. 1330-1342, Oct. 2012.
[28] Y. Y Wang, H. G. Zhang, J. Y. Zhang, and Y. C. Wang, "An SOS-based observer design for discrete-time polynomial fuzzy systems," *Int. J. Fuzzy Syst.*, vol.17, no. 1, pp. 94–104, 2015.
[29] T. V. Dang, W. J. Wang, L. Luoh, *et al.*, "Adaptive observer design for the uncertain Takagi–Sugeno fuzzy system with output disturbance," *IET Control Theory Appl.*, vol. 6, no. 12, pp. 1351–1366, 2012.
[30] W. J. Wang, V. P. Vu, W. Chang, C. H. Sun, and S. J. Yeh, "A synthesis of observer-based controller for stabilizing uncertain T-S fuzzy systems," *J. Intell. Fuzzy Syst.*, vol.30, pp. 3451–3463, 2016.
[31] H. H. Choi, "LMI-based nonlinear fuzzy observer-controller design for uncertain MIMO nonlinear system," *IEEE Trans. Fuzzy Syst.*, vol. 15, no.7, pp. 956–971, 2007.
[32] A. Golabi, M. Beheshti, M. H. Asemani, " H_∞ robust fuzzy dynamic observer-based controller for uncertain Takagi–Sugeno fuzzy systems," *IET Control Theory Appl.*, vol. 6, no. 12, pp. 1434–1444, 2012.
[33] T. V. Dang, W. J. Wang, C. H. Huang *et al*., "Observer synthesis for the T–S fuzzy system with uncertainty and output disturbance," *J. Intell. Fuzzy Syst.*, vol. 22, no. 6, pp. 173–183, 2011.
[34] H. S. Kim, J. B. Park, and Y. H. Joo, "Robust stabilization condition for a polynomial fuzzy system with parametric uncertainties," *the 2012 12th International Conference on Control, Automation and Systems*, Jeju Island, Korea, Oct., 2012, pp. 107-111.
[35] K. Tanaka, M. Tanaka, Y. J. Chen, and H. O. Wang, "A new sum-of-squares design framework for robust control of polynomial fuzzy systems with uncertainties," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 1, pp. 94-110, 2016.
[36] V. P. Vu and W. J. Wang, "Observer synthesis for uncertain Takagi–Sugeno fuzzy systems with multiple output matrices," *IET Control Theory Appl.*, vol. 10, no. 2, pp. 151–161, 2016.
[37] S. J. Yeh, W. Chang, and W.J Wang, "Unknown input based observer synthesis for uncertain T–S fuzzy systems," *IET Control Theory Appl.*, vol. 9, no.7, pp. 729–735, 2015.
[38] V. P. Vu and W. J. Wang, "Observer design for a discrete-time T-S fuzzy system with uncertainties," *the 2015 IEEE International Conference on Automation Science and Engineering (CASE)*, Gothenburg, Sweden, Aug. 2015, pp. 1262 - 1267.
[39] A. Chibani, M. Chadli, M. M. Belhaouane, and N. B. Braiek, "Polynomial observer design for unknown inputs polynomial fuzzy systems: A Sum of Squares approach," *the 53rd IEEE Conference on Decision and Control*, Los Angeles, California, USA, Dec., 2014, pp. 6788-6793.
[40] A. Chibani, M. Chadli, and N. B. Braiek, "A sum of squares approach for polynomial fuzzy observer design for polynomial fuzzy systems with unknown inputs," *International Journal of Control, Automation and Systems*, vol. 14, no. 1, pp. 323-330, 2016.
[41] K. Tanaka, T. Hori, and H. O. Wang, "A multiple Lyapunov function approach to stabilization of fuzzy control systems," *IEEE Trans. Fuzzy Syst.*, vol. 11, no. 4, pp. 582-589, 2003.
[42] S. H. Kim and P. G. Park, "Observer-based relaxed *H∞* control for fuzzy systems using a multiple Lyapunov function," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 2, pp. 476-484, 2009.
[43] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. Hoboken, NJ:Wiley, 2001.
[44] A. J. Laub, *Matrix Analysis for Scientists and Engineers*, Siam, 2005.
[45] P. J. Antsaklis and A. N. Michel, *Linear Systems*, chapter 7, Birkhauser Boston, 2006.
[46] *Polynomial Toolbox 3.0 Manual for Matlab*, PolyX, Ltd, September 2009.
[47] W.S. Levine, *The control hand book*, vol. 1, Jaico Publishing House, 1999.

Van-Phong Vu received the B.S. degree in electrical engineering from Ha Noi University of Sciences and Technology, Vietnam in 2007 and the M.S. degree in electrical engineering from Southern Taiwan University of Sciences and Technology, Taiwan in 2010. Since 2012, he has been a lecturer at Ho Chi Minh city, University of Education and Technology, Vietnam. He is currently pursuing the Ph.D. degree in electrical engineering at National Central University. His research interests are fuzzy systems, intelligent control, observer and controller design for uncertain system.

Wen-June Wang received the B.S. degree in the Department of Control Engineering from National Chiao-Tung University, Hsin-Chu, Taiwan in 1980; and M.S. degree in the Department of Electrical Engineering from Tatung University, Taipei, Taiwan in 1984. Moreover, he received the Ph.D. degree in the Institute of Electronics from National Chiao-Tung University of Taiwan in 1987. Prof. Wang is presently a Chair Professor of Department of Electrical Engineering. He was the Dean of College of Electrical Engineering and Computer Science, National Central University, Chung-Li City, Taiwan. He was also a Chair Professor and the Dean of the Research and Development Office of National Taipei University of Technology, Taipei, Taiwan in 2007~2009. In 2005~2007, he was the Dean of College of Science and Technology, National Chi-Nan University, Puli, NanTou, Taiwan. Moreover, Prof. Wang obtained the honor of *IEEE Fellow* in 2008 and *IFSA Fellow* in 2017. Prof. Wang has authored or coauthored over 160 refereed journal papers and 160 conference papers in the areas of fuzzy systems and theorems, robust and nonlinear control in large scale systems, and neural networks etc. His most significant contributions are the design of fuzzy systems and the development of robotics. His other research interests include the areas of robot control, neural networks, and pattern recognition etc.

Hsiang-Chieh Chen received his B.S. degree in engineering science from the National Cheng Kung University, Taiwan, in 2002 and his M.S. and Ph.D. degrees in electrical engineering from the National Central University, Taoyuan, Taiwan, in 2005 and 2009. Since Aug. 2016, he joined the faculty of the Department of Electrical Engineering, National United University, Miaoli, Taiwan, where he is currently an assistant professor. His research interests include image processing, computer vision, artificial intelligence, and robotics.

Jacek M. Zurada received the MS and PhD degrees in electrical engineering from the Technical University of Gdansk, Poland, in 1968 and 1975, respectively. He has published over 400 joural and conference papers in various areas. From 1998 to 2003, he was the editor-in-chief of the IEEE Transactions on Neural Networks. He was an associate editor of the IEEE Transactions on Circuits and Systems, Part I and Part II, and served on the editorial board of the Proceedings of IEEE. He is an associate editor of Neural Networks, Neurocomputing, Schedae Informaticae, and the International Journal of Applied Mathematics and Computer Science, the advisory editor of the International Journal of Information Technology and Intelligent Computing, and the editor of Springer's Natural Computing Book Series. He was elected as the President of the IEEE Computational Intelligence Society in 2004-2005. He now chairs the IEEE TAB Periodicals Committee (2010- 11), the IEEE TAB Periodicals Review Committee (2012-13), and a Life Fellow of the IEEE.