Cellular Neural Networks With Transient Chaos

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***Abstract*—A new model of cellular neural networks (CNNs) with transient chaos is proposed by adding negative self-feedbacks into CNNs after transforming the dynamic equation to discrete time via Euler's method. The simulation on the single neuron model shows stable fix points, bifurcation and chaos. Hence, this new CNN model has richer and more flexible dynamics, and therefore may possess better capabilities of solving various problems, compared to the conventional CNN with only stable dynamics.**

***Index Terms*—Bifurcation, cellular neural networks (CNNs), chaos.**

I. Introduction

CELLULAR neural networks (CNNs) [1] are especially suitable for very large-scale integration (VLSI) implementation because of local interconnectivities in CNNs. Some promising applications of CNNs in image processing, pattern recognition, and combinatorial optimization problems have been reported in [2]–[6].

A number of authors have studied nonlinear dynamics in CNNs [7]–[11]. Chen *et al.* [8] proposed a discrete-time CNN (DTCNN) with neuronal input–output described by the logistic function; however, this type of neurons are rarely used in the literature. In a CNN with time delays, Lu *et al.* [9] investigated complex dynamics by simply adding a piecewise-linear delayed feedback to the linear system. Zou and Nossek [10] have also found bifurcation and chaos in small autonomous networks with only two or three cells. Biey *et al.* [11] investigated equilibrium point bifurcation and other complex dynamics in a first-order autonomous space-invariant CNN.

Analog realizations of CNNs have been widely discussed [12]–[15]. For example, Harrer and Nossek [12] designed and implemented a high-density layout for the DTCNN. Chou *et al.* [13] proposed a mode chip with hardware annealing as a highly efficient method of finding globally optimal solutions with CNNs. Linan *et al.* [14] presented an analog programmable visual microprocessor chip, i.e., the ACE4k, which can work as a CNN universal machine [16] and process complex spatio-temporal images in parallel. Rodriguez-Vazquez *et al.* improved ACE chips as they designed and presented an ACE16k [15] with better performance, more functionalities, and lower power consumption.

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Many problems in science and technology are combinatorial optimization problems, with energy or cost functions to be minimized under some constraints. Extensive research effort has gone into developing efficient techniques for finding minimum values of energy functions. Since Hopfield and Tank's seminal work on solving the traveling salesman problem (TSP) with the Hopfield neural network (HNN) [17], the HNN has been widely used in optimization problems. However, the fact that the HNN can often be trapped at a local minimum discouraged attempts at solving various optimization problems with the HNN.

Several methods, such as stochastic simulated annealing (SSA) [18] which allows for temporary energy increases, have been incorporated into the HNN, in order to improve the solution quality. Chen and Aihara [19] proposed chaotic simulated annealing (CSA) by adding a negative self-feedback to a Hopfield-type neural network and gradually removing this negative self-feedback. This network leads to remarkable improvements over the HNN in terms of solution quality of optimization problems. Wang and Smith [20] suggested an alternative approach to CSA by reducing the time step rather than the self-feedback and generalized the stability theorems to less restrictive and more compact forms. Wang *et al.* [21] combined the SSA and CSA, and proposed a new approach, i.e., stochastic CSA (SCSA) for solving optimization problems. The simulation results on a TSP and a channel assignment problem demonstrated the effectiveness of SCSA.

When simulated in serial computers, such as a desk-top personal computer, neural networks are known to be slow and rather ineffective in solving optimization problems in comparisons with classical methods, e.g., branch-and-bound technique and tabu search. The advantages of neural networks can be truly realized only when the parallel processing capability of neural networks is used, e.g., when neural networks are implemented in hardware.

In order to take advantage of both efficient search with chaos [22]–[24] and the mature technology of VLSI implementation of CNNs [25]–[28], we propose a *transiently chaotic CNN* (TC-CNN). Compared to existing chaotic CNNs, our model uses conventional neurons and is relatively easy to analyze mathematically when the network is large (some results were reported at a recent conference [29]).

This brief is organized as follows. In Section II, the TC-CNN is proposed by adding negative self-feedback to the Euler approximation of the continuous CNN. Then the stability analysis of the network is presented. The simulation result of a single neuron model is given in Section III. Finally, we conclude this brief in Section IV.

II. TC-CNNs

In this section, we first briefly review the continuous CNN model. The TC-CNN is then proposed. In the last part of this section, we present the stability condition for the TC-CNN.

A. Continuous CNN

The system equation for an $M \times N$ CNN is [1]

$$C\frac{dx_{ij}(t)}{dt} = -\frac{1}{R_x}x_{ij}(t) + \sum_{C(k,l)\in N_r(i,j)} A(i,j;k,l)y_{kl}(t) + \sum_{C(k,l)\in N_r(i,j)} B(i,j;k,l)u_{kl} + I \quad (1)$$

where

x_{ij}	internal state of neuron (i,j);
y_{ij}	output of neuron (i,j);
u_{ij}	input of neuron (i,j);
$A(i,j;k,l)$	output feedback parameter;
$B(i,j;k,l)$	input control parameter;
$C(i,j)$	neuron (i,j);
$N_r(i,j)$	$\{(k,l)\mid \max\{\lvert k-i\rvert, \lvert l-j\rvert\} \le r, 1 \le k \le M, 1 \le l \le N\}$ The r-neighborhood of neuron (i,j);
I	independent voltage source;
C;	linear capacitor;
R_x	linear resistor.

The output function of neuron (i,j) is

$$y_{ij}(t) = \frac{1}{2}(|x_{ij}(t)+1| - |x_{ij}(t)-1|). \quad (2)$$

Constraint conditions are

$$|x_{ij}(0)| \le 1, \qquad 1 \le i \le M; \quad 1 \le j \le N$$
$$|u_{ij}| \le 1, \qquad 1 \le i \le M; \quad 1 \le j \le N.$$

Any cell in a CNN is connected only to its neighboring cells. This CNN property of nearest neighbor interactions makes CNNs much more amenable to VLSI implementation compared to general neural networks.

B. TC-CNN

Let us consider the difference equation version of (1) by Euler's method [30]

$$\begin{aligned} x_{ij}(t+1) &= \left(1-\frac{\Delta t}{CR_x}\right)x_{ij}(t) + \frac{\Delta t}{C}\sum_{C(k,l)\in N_r(i,j)} A(i,j;k,l)y_{kl}(t) \\ &\quad + \frac{\Delta t}{C}\sum_{C(k,l)\in N_r(i,j)} B(i,j;k,l)u_{kl} + \frac{\Delta t}{C}I - z[y_{ij}(t)-I_0] \\ &= px_{ij}(t) + \sum_{C(k,l)\in N_r(i,j)} a(i,j;k,l)y_{kl}(t) \\ &\quad + \sum_{C(k,l)\in N_r(i,j)} b(i,j;k,l)u_{kl} + i - z[y_{ij}(t)-I_0] \end{aligned} \quad (3)$$

where we have added a self-feedback for each neuron as indicated by the last term in (3)

$$\begin{aligned} p &= 1-\frac{\Delta t}{CR_x} \\ a(i,j;k,l) &= A(i,j;k,l)\frac{\Delta t}{C} \\ b(i,j;k,l) &= B(i,j;k,l)\frac{\Delta t}{C} \\ i &= I\frac{\Delta t}{C} \\ z &= \text{Self-feedback connection weight} \\ I_0 &= \text{A positive bias factor.} \end{aligned}$$

In the conventional CNN, there are usually positive self-feedbacks, i.e., $A(i,j,i,j) \ge 0$. When the feedback is positive, it has a stabilizing effects [31] and is equivalent to a hysteresis. When the feedback is negative, oscillations and chaos can occur, depending on the magnitude of the negative feedback, similar to the TCNN investigated by Chen and Aihara [19]. Unstable dynamic behaviors will occur if the magnitude of the negative self-feedback is sufficiently large. We may change the chaotic system to transiently chaotic system by gradually removing the negative self-feedback. The self-feedback can be reduced in any schedule, which may be designed for different problems. Here, we set it to decay exponentially

$$z(t+1) = (1-\beta)z(t) \quad (4)$$

where $\beta (0 \le \beta \le 1)$ is the damping factor of the time-dependent self-feedback $z(t)$.

Combining (3) with the activation function and the self-feedback decaying function, we obtain the definition of the new model

$$x_{ij}(t+1) = px_{ij}(t) + \sum_{C(k,l)\in N_r(i,j)} a'(i,j;k,l)y_{kl}(t) + \sum_{C(k,l)\in N_r(i,j)} b(i,j;k,l)u_{kl} + i' \quad (5)$$

$$y_{ij}(t) = \frac{1}{2}(|x_{ij}(t)+1| - |x_{ij}(t)-1|) \quad (6)$$

$$z(t+1) = (1-\beta)z(t) \quad (7)$$

where

$$a'(i,j;i,j) = a(i,j;i,j) - z(t) \quad (8)$$

$$i' = i + z(t)I_0. \quad (9)$$

To compare with the continuous-time CNN, the new TC-CNN model in (5)–(7) would correspond to the following differential equations:

$$\frac{dx_{ij}(t)}{dt} = -z_{ij}(t)(y_{ij}(t)-I_0) + px_{ij}(t) + \sum_{C(k,l)\in N_r(i,j)} A(i,j;k,l)y_{kl}(t) + \sum_{C(k,l)\in N_r(i,j)} B(i,j;k,l)u_{kl} + I \quad (10)$$

$$y_{ij}(t) = \frac{1}{2}(|x_{ij}(t)+1| - |x_{ij}(t)-1|) \tag{11}$$

$$\frac{dz_{ij}}{dt} = -\beta z_{ij}. \tag{12}$$

C. Stability of the TC-CNN

We use the following energy function for the TC-CNN:

$$E(t) = -\frac{1}{2}\sum_{(i,j)}\sum_{(k,l)} a'(i,j;k,l)y_{ij}(t)y_{kl}(t) - \sum_{(i,j)}\sum_{(k,l)} b(i,j;k,l)y_{ij}(t)u_{kl} - \sum_{(i,j)} i'y_{ij}(t) + \frac{1-p}{2}\sum_{(i,j)} y_{ij}(t)^2. \tag{13}$$

The first three terms are from the energy function for conventional continuous-time CNNs [1]. We modified the last term according to [32], [20] for the new discrete-time model.

The difference between two time steps in the energy function is

$$\begin{aligned}\Delta E = &-\frac{1}{2}\sum_{(i,j)}\sum_{(k,l)} a'(i,j;k,l)\Delta y_{ij}(t)\Delta y_{kl}(t) \\ &- \sum_{(i,j)}\sum_{(k,l)} a'(i,j;k,l)y_{kl}(t)\Delta y_{ij}(t) \\ &- \sum_{(i,j)} \Delta y_{ij}(t)\left[\sum_{(k,l)} b(i,j;k,l)u_{kl} + i'\right] \\ &+ \frac{1-p}{2}\sum_{(i,j)} \Delta y_{ij}(t)[y_{ij}(t+1)+y_{ij}(t)].\end{aligned} \tag{14}$$

According to the cell circuit (5), (14) can be written as

$$\begin{aligned}\Delta E = &-\frac{1}{2}\sum_{(i,j)}\sum_{(k,l)} a'(i,j;k,l)\Delta y_{ij}(t)\Delta y_{kl}(t) \\ &- \sum_{(i,j)} \Delta y_{ij}(t)[x_{ij}(t+1) - p\,x_{ij}(t)] \\ &+ \frac{1-p}{2}\sum_{(i,j)} \Delta y_{ij}(t)[y_{ij}(t+1)+y_{ij}(t)].\end{aligned} \tag{15}$$

From the output function given in (6), we have

$$y_{ij}(t) = x_{ij}(t), \quad \text{when } |x_{ij}(t)|<1$$
$$\Delta y_{ij}(t) = 0, \quad \text{when } |x_{ij}(t)| \geq 1$$

then

$$\begin{aligned}\Delta E = &-\frac{1}{2}\sum_{|x_{ij}|<1}\sum_{|x_{kl}|<1} a'(i,j;k,l)\Delta y_{ij}(t)\Delta y_{kl}(t) \\ &- \sum_{|x_{ij}|<1} \Delta y_{ij}(t)[y_{ij}(t+1) - p\,y_{ij}(t)] \\ &+ \frac{1-p}{2}\sum_{|x_{ij}|<1} \Delta y_{ij}(t)[y_{ij}(t+1)+y_{ij}(t)] \\ = &-\frac{1}{2}\sum_{|x_{ij}|<1}\sum_{|x_{kl}|<1} [a'(i,j;k,l) + (1+p)\delta_{ik}\delta_{jl}] \\ &\times \Delta y_{ij}(t)\Delta y_{kl}(t).\end{aligned} \tag{16}$$

Therefore, according to Lyapunov theorem, if the matrix $[a'(i,j;k,l) + (1+p)\delta_{ik}\delta_{jl}]$ is positive-definite, i.e., $\Delta E \leq 0$, then the network is stable. Hence, a sufficient stability condition for the TC-CNN is

$$a'(i,j;i,j) > -(1+p) = \frac{\Delta t}{CR_x} - 2 \tag{17}$$

where CR_x is the time constant of the dynamics of the circuit [1]. Substituting the (8) into (17)

$$a(i,j;i,j) - z(t) > -(1+p). \tag{18}$$

If $p = 0.7$, based on (17), the system is stable and converges to a fixed point when

$$a(i,j;i,j) - z(t) > -(1+p) = -1.7. \tag{19}$$

These facts are verified by the simulation result shown in Section III.

III. Transiently Chaotic Dynamics of Single Neuron Model

The single neuron model obtained from (3)–(4) is

$$\begin{aligned}x(t+1) = &-z(t)(y(t) - I_0) + p\,x(t) \\ &+ Ay(t) + B\,u + I\end{aligned} \tag{20}$$

$$y(t) = \frac{1}{2}(|x(t)+1| - |x(t)-1|) \tag{21}$$

$$z(t+1) = (1-\beta)z(t). \tag{22}$$

Let $Bu + I = I'$. The one-dimensional mapping function from $x(t)$ to $x(t+1)$ becomes, with $z(t)$ fixed at z_0 as shown in (23), at the bottom of the page.

Fig. 1 presents an example of the one-dimensional mapping function from $x(t)$ to $x(t+1)$ when $p = 0.7, A = 0.1, I' = 0.08, I_0 = 0.35, z_0 = 5$; It shows that the slope of the mapping

$$x(t+1) = \begin{cases} -z_0(1-I_0) + p\,x(t) + A + I', & 1 \leq x(t) \\ -z_0(x(t) - I_0) + (p+A)\,x(t) + I', & -1 \leq x(t) \leq 1 \\ -z_0(-1-I_0) + px(t) - A + I', & x(t) \leq -1 \end{cases} \tag{23}$$

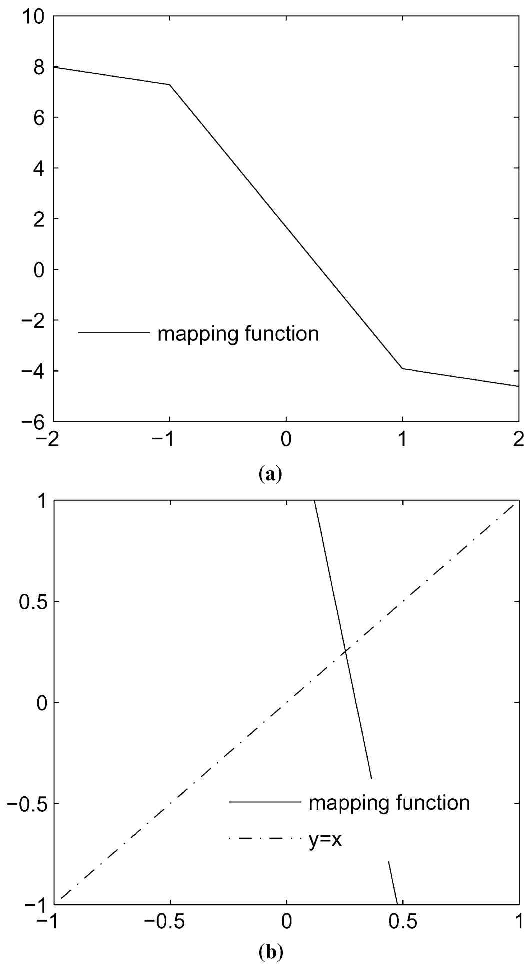

Fig. 1. One-dimensional mapping function from $x(t)$ to $x(t+1)$ for the TC-CNN. (a) Zoom out. (b) Zoom in.

function at the intersection is less than -1, which implies that the system is unstable [33].

To simulate the single neuron model, we choose parameters as follows for (20), $A = 0.1, B = -1.8, p = 0.7, I = 0.1, I_0 = 0.35, u = 0.1, \beta = 0.001, z(1) = 5, x(1)$ is random number whose element is distributed in the interval (0,1). The result in the Fig. 2 shows the time evolutions of $y(t), z(t)$ and the Lyapunov exponent λ of $y(t)$. With exponential damping of $z(t)$, the dynamics of the network transit from chaos to periodic attractors and then fixed point through reversed discontinuity induced bifurcations [34]. It is different from the reversed period-doubling bifurcations presented by Chen and Aihara [19], as Chen and Aihara used a nonlinear output function whereas CNNs have a piece-wise linear output function [35].

Lyapunov exponent [36] is a crucial index to identify deterministic chaos. The definition we used is

$$\lambda = \lim_{m \to +\infty} \frac{1}{m} \sum_{t=0}^{m-1} \ln \left| \frac{dx(t+1)}{dx(t)} \right|. \tag{24}$$

A positive λ indicates chaos. Fig. 2(c) presents the relationship between the Lyapunov exponent and the negative self-feedback parameter z. The Lyapunov exponent is positive $(\lambda > 0)$ when the self-feedback parameter $z > 2.3$ (or when $t < 500$). In the meantime, we can see the chaotic behavior clearly in the

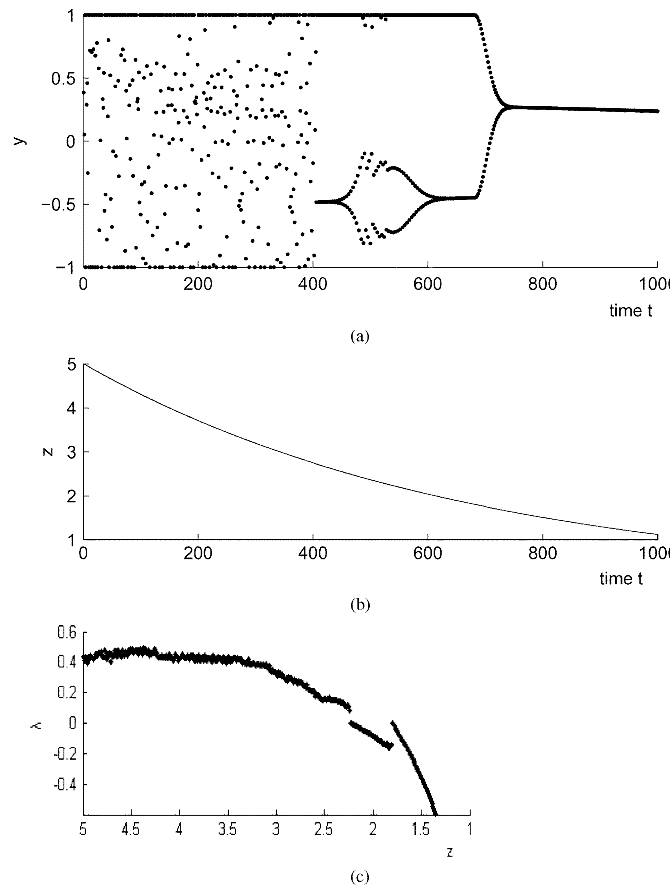

Fig. 2. Dynamics of the single neuron model. (a) Output of the neuron. (b) Self-feedback in network. (c) Relationship between the Lyapunove exponent and the self-feedback.

Fig. 2(a) in this domain. We have also simulated a two-dimensional 3×3 network, and it shows identical bifurcation diagrams with the single neuron model.

IV. Conclusion

In this brief, we proposed a discrete-time TC-CNN. We showed that the Euler approximation of the continuous-time CNNs may have complex dynamics when we add negative self-feedbacks, depending on the magnitude of the self-feedback. The length of the chaotic dynamic can be determined by the initial value of the self-feedback $z(0)$ and the damping factor β. The network may benefit more from the ergodic nature of chaos if the chaotic transient last longer, i.e., a larger $z(0)$ or a smaller β; However, the network will need more time to converge and find a solution for an application. More study need to be done to analyze the relationship between the performance of TC-CNN model and its chaotic duration. Different applications may require different lengths of chaotic dynamic and different types of damping scheme (we used exponential damping in this brief).

An analog circuit structure and a layout for the realization of DTCNNs were introduced in [12], the algorithm is

$$x_{ij}(t+1) = \sum_{C(k,l) \in N_r(i,j)} A(i,j;k,l) y_{kl}(t) + \sum_{C(k,l) \in N_r(i,j)} B(i,j;k,l) u_{kl} + I_{ij}. \tag{25}$$

Contrasting the new model (3) with (25), we see the following two differences between the TC-CNN model and DTCNNs: firstly, the new model has a new term $p\,x_{ij}(t)$; secondly, there should be negative self-feedback in the new model, and the value needs to damp exponentially. As the $x_{ij}(t)$ represents the internal state of the neuron, the current $p\,x_{ij}(t)$ can be implemented through a buffer and an amplifier with a gain of p. Furthermore, $\sum a(i,j;k,l)y_{kl}(t)$ is implemented by the current supply in the analog implementation of DTCNNs, and the positive or negative of the value is depend on the direction of the current supply. On the other hand, the damping of the value can be achieved by adjusting the parameters of the switched-capacitor. Therefore, our new model can be easily implemented in hardware.

The new *TC-CNN* has two distinctive properties. It has complex dynamics and therefore better ability to search for optimal solutions. Furthermore, the local connectivity of the TC-CNN makes it especially suited for large scale circuit implementation.

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